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ANALYSE DES INFINIMENT-PETITS
CALCUL DIFERENTIEL
TABLE DES SINUS
De la Monce inv.
Soubeyran Sculp

INTRODUCTIO
IN ANALYSIN
INFINITORUM.

AUCTORE

LEONHARDO EULERO,

Professore Regio BEROLINENSI, *& Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.



LAUSANNÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.



A Lausanne et Geneve, chez MARÇ-MICHEL BOUSQUET *et Comp.^e 1748.*



ILLUSTRISSIMO VIRO

JOHANNI JACOBO

DORTOUS DE MAIRAN,

UNI EX XLVIRIS

ACADEMIÆ GALLICÆ,

REGIÆ ETIAM SCIENTIARUM

PARISIENSIS,

IN QUA SECRETARII PERPETUI MUNUS NUPER

ABDICAVIT,

*   NEC

NEC NON

ALIARUM BENE MULTARUM,

LONDINENSIS, PETROPOLITANÆ,

&c.

SOCIETATUM, ACADEMIARUMVE

SOCIO DIGNISSIMO.

MARCUS-MICHAEL BOUSQUET.

VIR ILLUSTRISSIME,

P*Atronos* Euleriano *scripto quærere necesse neutiquam esse, Mathematicarum Disciplinarum cultoribus satis constat. Sciunt utique illi, varias earum partes novis eum luminibus sic illustrasse, ut indè meritò claris-simi*

ſimi rerum in his abſtruſiſſimarum interpretis locum ſit conſequutus. Quem quin egregiè tueatur, immò tollat ſe altiùs quoque opere iſthoc, nemo dubitabit, certior hiſce factus, indulſiſſe Te mihi, ut illuſtriſſimo nomini Tuo dicatum publicè prodiret. Pertinere autem hoc in me collatum beneficium ad Auctoris decus probe intelligens, Ipſe, ut eo uterer, lubens conceſſit; & cum in rem meam faciat omnimodò, quî neglexiſſem?

Ab his equidem, quibus Libros inſcribunt, ſibi neſcio quid ideò deberi, plerique tacitè conſtituunt; acceptaque beneficia quodammodo remunerari, ut ſeſe ferè nexu liberent omni. Ego verò ſecus ſentio. Mihi certè merum eſt beneficium patroni, quòd ſcriptoris aut excuſoris

opera id genus honore condecorari patiantur. Hac mente utique Tibi, VIR ILLUSTRISSIME, *animi gratissimi summæque observantiæ professionem hisce publicam excipias, rogo.*

Paratum promtumque semper juvandis litterarum studiis qui Te *novit, & notus vel hoc nomine es cuicumque in Republica doctorum Europæ totius non hospiti, plurimis officiis meæ etiam conditionis homines à* Te *affectos fuisse statuat necesse est. Nempe, tanquam* Tibi *uni esset injunctum curare, ut floreant humanum ingenium illustrantes scientiæ omnes, hominumque in usus adinventæ artes, ad singulis inservientium artifices etiam* Te *demittere dignaris, vel ab illa sublimium rerum perscrutatione, Cælive ip-*

sius

sius Tibi *tam nota regione, ut quæ hucusque mentes hominum metu complebant Phænomena minus intellecta, per* Te *jam grato tantùm admirationis sensu contemplentur, earumque causas habeant perspectas.*

Hinc ille veluti ex condicto Academiarum Orbis eruditi concursus, ut adlectum Te *cætui suo consequerentur, ornamento aliàs carituro insigni, quo cæteras nollent præ se frui. Hinc inprimis Illustrissimæ Parisiensis de* Te *judicium, cum ageretur de successore sufficiendo in locum emeriti* Fontenellii, *Viri, cujus ex ore calamoque fluere Scientiarum Artiumque omnium exquisitiores divitiæ, elegantiæque universæ perpetuo visæ sunt, & videbuntur dum sani sensus quicquam hu-*

mano ingenio erit. Tibi, *ſcilicet*, *Commentariorum Academiæ conſcribendorum provincia, cui præfectus ille erat, demandabatur continuò; quam, ut ornare diutiùs voluiſſes, docti omnes optabant: hoc uno minùs dolentes* Te *aliter cenſuiſſe, quòd aliis* Tibi *magis placituris, profuturisque nihilominus litteris in univerſum eruditionis ingeniive theſauros impenderes. Quod ut ad ultimas uſque metas hominum vitæ poſitas incolumis, florens, atque beatus præſtes, omni votorum contentione precor. Vale!*

Dabam *Lauſannæ* die 1. Aprilis
Anni Æræ Dionyſ. 1748.

PRÆFATIO.

SÆpenumero animadverti, maximam difficultatum partem, quas Matheſeos cultores in addiſcenda Analyſi infinitorum offendere ſolent, inde oriri, quod, Algebra communi vix apprehenſa, animum ad illam ſublimiorem artem appellant; quo fit, ut non ſolum quaſi in limine ſubſiſtant, ſed etiam perverſas ideas illius infiniti, cujus notio in ſubſidium vocatur, ſibi forment. Quanquam autem Analyſis infinitorum non perfectam Algebræ communis, omniumque artificiorum adhuc inventorum cognitionem requirit; tamen plurimæ extant quæſtiones, quarum evolutio diſcentium animos ad ſublimiorem ſcientiam præparare valet, quæ tamen in communibus Algebræ elementis, vel omittuntur, vel non ſatis accurate pertractantur. Hanc ob rem non dubito, quin ea, quæ in his libris congeſſi, hunc defectum abunde ſupplere queant. Non ſolum enim operam dedi, ut eas res, quas Analyſis infini-

infinitorum abſolute requirit, uberius atque diſtinctius exponerem, quam vulgo fieri ſolet; ſed etiam ſatis multas quæſtiones enodavi, quibus Lectores ſenſim & quaſi præter expectationem ideam infiniti ſibi familiarem reddent. Plures quoque quæſtiones per præcepta communis Algebræ hic reſolvi, quæ vulgo in Analyſi infinitorum tractantur: quo facilius deinceps utriuſque Methodi ſummus conſenſus eluceat.

Diviſi hoc Opus in duos Libros, in quorum priori, quæ ad meram Analyſin pertinent, ſum complexus: in poſteriori vero, quæ ex Geometria ſunt ſcitu neceſſaria, explicavi, quoniam Analyſis infinitorum ita quoque tradi ſolet, ut ſimul ejus applicatio ad Geometriam oſtendatur. In utroque autem prima Elementa prætermiſi, eaque tantum exponenda duxi, quæ alibi, vel omnino non, vel minus commode tractata, vel ex diverſis principiis petita reperiuntur.

In primo igitur Libro, cum univerſa Analyſis infinitorum circa quantitates variabiles earumque Functiones verſetur, hoc argumentum de Functionibus inprimis fuſius expoſui; atque Functionum tam transformationem, quam reſolutionem & evolutionem per ſeries infinitas demonſtravi. Complures enumeravi Functionum ſpecies, quarum in Analyſi ſublimiori præcipue ratio eſt habenda. Primum eas diſtinxi in algebraicas & tranſcendentes; quarum illæ per operationes in Algebra communi uſitatas ex quantitatibus variabilibus formantur, hæ vero vel per alias rationes componuntur, vel ex iiſdem operationibus infinities repetitis efficiuntur. Algebraicarum functionum primaria ſubdiviſio fit in rationales & irrationales, priores docui cum in partes ſimpliciores, tum in factores reſolvere; quæ operatio in Calculo integrali maximum adjumentum affert; poſteriores vero, quemadmodum idoneis ſubſtitutionibus ad formam rationalem perduci queant oſtendi. Evolutio autem per ſeries infinitas ad utrumque genus æque pertinet, atque etiam ad Functiones

ctiones transcendentes summa cum utilitate applicari solet; at quantopere doctrina de seriebus infinitis Analysin subliniorem amplificaverit, nemo est qui ignoret. Nonnulla igitur adjunxi Capita, quibus plurium serierum infinitarum proprietates, atque summas sum scrutatus; quarum quædam ita sunt comparatæ, ut sine subsidio Analysis infinitorum vix investigari posse videantur. Hujusmodi series sunt, quarum summæ exprimuntur, vel per Logarithmos vel Arcus circulares: quæ quantitates cum sint transcendentes, dum per quadraturam Hyperbolæ & Circuli exhibentur, maximam partem demum in Analysi infinitorum tractari sunt solitæ. Postquam autem a potestatibus ad quantitates exponentiales essem progressus, quæ nil aliud sunt nisi potestates, quarum exponentes sunt variabiles; ex earum conversione maxime naturalem ac fœcundam Logarithmorum ideam sum adeptus: unde non solum amplissimus eorum usus sponte est consecutus, sed etiam ex ea cunctas series infinitas, quibus vulgo istæ quantitates repræsentari solent, elicere licuit: hincque adeo facillimus se prodidit modus Tabulas Logarithmorum construendi. Simili modo in contemplatione Arcuum circularium sum versatus; quod quantitatum genus, etsi a Logarithmis maxime est diversum, tamen tam arcto vinculo est connexum, ut dum alterum imaginarium fieri videtur, in alterum transeat. Repetitis autem ex Geometria quæ de inventione Sinuum & Cosinuum Arcuum multiplorum ac submultiplorum traduntur, ex Sinu vel Cosinu cujusque Arcus expressi Sinum Cosinumque Arcus minimi & quasi evanescentis, quo ipso ad series infinitas sum deductus: unde, cum Arcus evanescens Sinui suo sit æqualis, Cosinus vero radio, quemvis Arcum cum suo Sinu & Cosinu ope serierum infinitarum comparavi. Tum vero tam varias expressiones cum finitas tum infinitas pro hujus generis quantitatibus obtinui, ut ad earum naturam perspiciendam Calculo infinitesimali prorsus non amplius

effet opus. Atque quemadmodum Logarithmi peculiarem Algorithmum requirunt, cujus in universa Analysi summus extat usus, ita quantitates circulares ad certam quoque Algorithmi normam perduxi; ut in calculo æque commode ac Logarithmi & ipsæ quantitates algebraicæ tractari possent. Quantum autem hinc utilitatis ad resolutionem difficillimarum quæstionum redundet, cum nonnulla Capita hujus Libri luculenter declarant, tum ex Analysi infinitorum plurima specimina proferri possent, nisi jam satis essent cognita, & indies magis multiplicarentur. Maximum autem hæc investigatio attulit adjumentum ad Functiones fractas in factores reales resolvendas; quod argumentum, cum in Calculo integrali sit prorsus necessarium, diligentius enucleavi. Series postmodum infinitas, quæ ex hujusmodi Functionum evolutione nascuntur, & quæ recurrentium nomine innotuerunt, examini subjeci; ubi earum tam summas quam terminos generales, aliasque insignes proprietates exhibui: & quoniam ad hæc resolutio in factores manuduxit, ita vicissim, quemadmodum producta ex pluribus, imo etiam infinitis, factoribus conflata per multiplicationem in series explicentur, perpendi. Quod negotium non solum ad cognitionem innumerabilium serierum viam aperuit, sed quia hoc modo series in producta ex infinitis factoribus constantia resolvere licebat, satis commodas inveni expressiones numericas, quarum ope Logarithmi Sinuum, Cosinuum, & Tangentium facillime supputari possunt. Præterea quoque ex eodem fonte solutiones plurium quæstionum, quæ circa partitionem numerorum proponi possunt, derivavi; cujusmodi quæstiones sine hoc subsidio vires Analyseos superare videantur. Hæc tanta materiarum diversitas in plura volumina facile excrescere potuisset; sed omnia, quantum fieri potuit, tam succincte proposui, ut ubique fundamentum clarissime quidem explicaretur, uberior vero amplificatio industriæ Lectorum relinqueretur; quo habeant, quibus

quibus vires ſuas exerceant, fineſque Analyſeos ulterius promoveant. Neque enim vereor profiteri, in hoc Libro non ſolum multa plane nova contineri; ſed etiam fontes eſſe detectos, unde plurima inſignia inventa adhuc hauriri queant.

Eodem inſtituto ſum uſus in altero Libro, ubi, quæ vulgo ad Geometriam ſubliomiorem referri ſolent, pertractavi. Antequam autem de Sectionibus Conicis, quæ alias fere ſolæ hunc locum occupant, agerem; Theoriam Linearum Curvarum in genere ita propoſui, ut ad ſcrutationem naturæ quarumvis Linearum Curvarum cum utilitate adhiberi poſſet. Ad hoc nullum aliud ſubſidium affero, præter æquationem, qua cujuſque Lineæ Curvæ natura exprimitur; ex eaque cum figuram, tum primarias proprietates deducere doceo: id quod potiſſimum in Sectionibus Conicis præſtitiſſe mihi ſum viſus; quæ antehac vel ſecundum ſolam Geometriam vel per Analyſin quidem, ſed nimis imperfecte ac minus naturaliter, tractari ſunt ſolitæ. Ex æquatione ſcilicet generali pro Lineis ſecundi ordinis primum earum proprietates generales explicavi, tum eas in genera ſeu ſpecies ſubdiviſi; reſpiciendo utrum habeant ramos in infinitum excurrentes, an vero tota Curva finito ſpatio includatur. Priori autem caſu inſuper diſpiciendum erat, quot ſint rami in infinitum excurrentes, & cujus naturæ ſint ſinguli; an habeant Lineas rectas aſymtotas, an minus. Sicque obtinui tres conſuetas Sectionum Conicarum ſpecies; quarum prima eſt Ellipſis, tota in ſpatio finito contenta; ſecunda autem Hyperbola, quæ quatuor habet ramos infinitos ad duas rectas aſymptotas convergentes; tertia vero ſpecies prodiit Parabola duos habens ramos infinitos aſymtotis deſtitutos. Simili porro ratione Lineas tertii ordinis ſum perſecutus, quas, poſt expoſitas earum proprietates generales, diviſi in ſedecim genera; ad eaque omnes ſeptuaginta duas ſpecies NEWTONI revocavi. Ipſam vero methodum ita clare deſcripſi, ut pro

quovis Linearum ordine ſequente diviſio in genera facilli-me inſtitui queat; cujus negotii periculum quoque ſeci in Lineis quarti ordinis. His deinde, quæ ad ordines Linearum pertinent, expeditis, reverſus ſum ad generales omnium Linearum affectiones eruendas. Explicavi itaque methodum definiendi tangentes curvarum, earum normales, atque etiam ipſam curvaturam, quæ per radium oſculi æſtimari ſolet: quæ etſi nunc quidem plerumque Calculo differentiali abſolvuntur, tamen idem per ſolam communem Algebram hic præſtiti, ut deinceps tranſitus ab Analyſi finitorum ad Analyſin infinitorum eo facilior reddatur. Perpendi etiam curvarum puncta flexus contrarii, cuſpides, puncta duplicia, ac multiplicia; modumque expoſui hæc omnia ex æquationibus ſine ulla difficultate definiendi. Interim tamen non nego, has quæſtiones multo facilius Calculi differentialis ope enodari poſſe. Attigi quoque controverſiam de cuſpide ſecundi ordinis, ubi ambo arcus in cuſpidem coeuntes curvaturam in eandem partem vertunt; eamque ita compoſuiſſe mihi videor, ut nullum dubium amplius ſupereſſe poſſit. Denique adjunxi aliquot Capita, in quibus Lineas Curvas, quæ datis proprietatibus gaudeant, invenire docui; pluraque tandem Problemata circa ſingulares Circuli ſectiones ſoluta dedi. Quæ cum ſint ea ex Geometria, quæ ad Analyſin infinitorum addiſcendam maximum adminiculum afferre videntur, Appendicis loco ex Stereometria Theoriam ſolidorum eorumque ſuperficierum per Calculum propoſui, & quemadmodum cujuſque ſuperficiei natura per æquationem inter tres variabiles exponi queat, oſtendi. Hinc, ſuperficiebus inſtar linearum in ordines digeſtis, ſecundum dimenſionum quas variabiles in æquatione conſtituunt numerum, in primo ordine ſolam ſuperficiem planam contineri oſtendi. Superficies vero ſecundi ordinis, ratione habita partium in infinitum expanſarum, in ſex genera diviſi; ſimilique modo pro ceteris ordinibus diviſio inſtitui poterit.

poterit. Contemplatus ſum quoque interſectiones duarum ſuperficierum; quæ cum plerumque ſint curvæ non in eodem plano ſitæ, quemadmodum æquationibus comprehendi queant, monſtravi. Tandem etiam poſitionem planorum tangentium, atque rectarum, quæ ad ſuperficies ſint normales, determinavi.

De cetero, cum non paucæ res hic occurrant ab aliis jam tractatæ, veniam rogare me oportet, quod non ubique honorificam mentionem eorum, qui ante me in eodem genere elaborarunt, fecerim. Cum enim mihi propoſitum eſſet omnia quam breviſſime pertractare, Hiſtoria cujuſque Problematis magnitudinem operis non mediocriter auxiſſet. Interim tamen pleræque quæſtiones, quæ alibi quoque ſolutæ reperiuntur, hic ſolutiones ex aliis principiis ſunt nactæ; ita ut non exiguam partem mihi vindicare poſſem. Spero autem cum iſta, tum ea potiſſimum, quæ prorſus nova hic proferuntur, pleriſque, qui hoc ſtudio delectantur, non ingrata eſſe futura.

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INTRODUCTIO
IN
ANALYSIN INFINITORUM.
LIBER PRIMUS,

Continens

Explicationem de Functionibus quantitatum variabilium; earum resolutione in Factores, atque evolutione per Series infinitas: una cum doctrina de Logarithmis, Arcubus circularibus, eorumque Sinubus & Tangentibus; pluribusque aliis rebus, quibus Analysis infinitorum non mediocriter adjuvatur.



LIBER PRIMUS.

CAPUT PRIMUM.

De Functionibus in genere.

1. *Quantitas conſtans eſt quantitas determinata, perpetuo eumdem valorem ſervans.*

Ejuſmodi quantitates ſunt numeri cujuſvis generis, quippe qui eumdem, quem ſemel obtinuerunt, valorem conſtanter conſervant: atque ſi hujuſmodi quantitates conſtantes per characteres indicare convenit, adhibentur litteræ Alphabethi initiales a, b, c, &c. In Analyſi quidem communi, ubi tantum quantitates determinatæ conſiderantur, hæ litteræ Alphabethi priores quantitates cognitas denotare ſolent, poſteriores vero quantitates incognitas; at in Analyſi ſublimiori hoc diſcrimen non tantopere ſpectatur, cum hic ad illud quantitatum diſcrimen præcipue reſpiciatur, quo aliæ conſtantes, aliæ vero variabiles ſtatuuntur.

LIB. I. 2. *Quantitas variabilis est quantitas indeterminata seu universalis, quæ omnes omnino valores determinatos in se complectitur.*

Cum ergo omnes valores determinati numeris exprimi queant, quantitas variabilis omnes numeros cujusvis generis involvit. Quemadmodum scilicet ex ideis individuorum formantur ideæ specierum & generum; ita quantitas variabilis est genus, sub quo omnes quantitates determinatæ continentur. Hujusmodi autem quantitates variabiles per litteras Alphabethi postremas z, y, x, &c. repræsentari solent.

3. *Quantitas variabilis determinatur, dum ei valor quicunque determinatus tribuitur.*

Quantitas ergo variabilis innumerabilibus modis determinari potest, cum omnes omnino numeros ejus loco substituere liceat. Neque significatus quantitatis variabilis exhauritur, nisi omnes valores determinati ejus loco fuerint substituti. Quantitas ergo variabilis in se complectitur omnes prorsus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales quam irrationales & transcendentes. Quinetiam cyphra & numeri imaginarii a significatu quantitatis variabilis non excluduntur.

4. *Functio quantitatis variabilis, est expressio analytica quomodocunque composita ex illa quantitate variabili, & numeris seu quantitatibus constantibus.*

Omnis ergo expressio analytica, in qua præter quantitatem variabilem z omnes quantitates illam expressionem componentes sunt constantes, erit Functio ipsius z: Sic $a + 3z$; $az - 4zz$; $az + b\sqrt{(aa - zz)}$; c^z; &c. sunt Functiones ipsius z.

5. *Functio ergo quantitatis variabilis ipsa erit quantitas variabilis.*

Cum enim loco quantitatis variabilis omnes valores determinatos substituere liceat, hinc Functio innumerabiles valores determinatos induet; neque ullus valor determinatus excipietur, quem Functio induere nequeat, cum quantitas variabilis quoque valores imaginarios involvat. Sic etsi hæc Functio $\sqrt{(9 - zz)}$, numeris realibus loco z substituendis, nunquam valorem ternario majorem recipere potest; tamen ipsi z valores imaginarios tribuendo

tribuendo ut $5\sqrt{-1}$, nullus aſſignari poterit valor determinatus quin ex formula $\sqrt{(9-zz)}$ elici queat. Occurrunt autem nonnunquam Functiones tantum apparentes, quæ, utcunque quantitas variabilis varietur, tamen uſque eumdem valorem retinent, ut z^0; 1^z; $\frac{aa-az}{a-z}$, quæ, etſi ſpeciem Functionis mentiuntur, tamen revera ſunt quantitates conſtantes.

6. *Præcipuum Functionum diſcrimen in modo compoſitionis, quo ex quantitate variabili & quantitatibus conſtantibus formantur, poſitum eſt.*

Pendet ergo ab Operationibus quibus quantitates inter ſe componi & permiſceri poſſunt: quæ Operationes ſunt Additio & Subtractio; Multiplicatio & Diviſio: Evectio ad Poteſtates & Radicum Extractio; quo etiam Reſolutio Æquationum eſt referenda. Præter has Operationes, quæ algebraïcæ vocari ſolent, dantur complures aliæ tranſcendentes, ut Exponentiales, Logarithmicæ, atque innumerabiles aliæ, quas Calculus integralis ſuppeditat.

Interim ſpecies quædam Functionum notari poſſunt; ut multipla $2z$; $3z$; $\frac{3}{5}z$; az; &c. & Poteſtates ipſius z, ut z^2; z^3; $z^{\frac{1}{2}}$; z^{-1}; &c. quæ, uti ex unica operatione ſunt deſumtæ, ita expreſſiones quæ ex operationibus quibuſcunque naſcuntur, Functionum nomine inſigniuntur.

7. *Functiones dividuntur in Algebraïcas & Tranſcendentes; illæ ſunt, quæ componuntur per operationes algebraïcas ſolas, hæ vero in quibus operationes tranſcendentes inſunt.*

Sunt ergo multiplæ ac Poteſtates ipſius z Functiones algebraïcæ; atque omnes omnino expreſſiones, quæ per operationes algebraïcas ante memoratas formantur, cujuſmodi eſt $\frac{a+bz^n-c\sqrt{(2z-zz)}}{aaz-3bz^3}$. Quin-etiam Functiones algebraïcæ ſæpenumero nequidem explicite exhiberi poſſunt, cujuſmodi Functio ipſius z eſt Z, ſi definiatur per hujuſmodi æquationem; $Z^5 = azzZ^3 - bz^4Z^2 + cz^3Z - 1$. Quanquam enim hæc

LIB. I. æquatio reſolvi nequit; tamen conſtat Z æquari expreſſioni cuipiam ex variabili z & conſtantibus compoſitæ; ac propterea fore Z Functionem quamdam ipſius z. Cæterum de Functionibus tranſcendentibus notandum eſt, eas demum fore tranſcendentes, ſi operatio tranſcendens non ſolum ingrediatur, ſed etiam quantitatem variabilem afficiat. Si enim operationes tranſcendentes tantum ad quantitates conſtantes pertineant, Functio nihilominus algebraïca eſt cenſenda: uti ſi c denotet circumferentiam Circuli, cujus radius ſit $= 1$, erit utique c quantit. tranſcendens, verumtamen hæ expreſſiones $c + z$; cz^2; $4z^c$ &c. erunt Functiones algebraïcæ ipſius z. Parvi quidem eſt momenti dubium quod a quibuſdam movetur, utrum ejuſmodi expreſſiones z^c Functionibus algebraïcis annumerari jure poſſint, necne; quinetiam Poteſtates ipſius z, quarum exponentes ſint numeri irrationales, uti $z^{\sqrt{2}}$ nonnulli maluerunt Functiones interſcendentes quam algebraïcas appellare.

8. *Functiones algebraïcæ ſubdividuntur in Rationales & Irrationales: illæ ſunt, ſi quantitas variabilis in nulla irrationalitate involvitur; hæ vero, in quibus ſigna radicalia quantitatem variabilem afficiunt.*

In Functionibus ergo rationalibus aliæ operationes præter Additionem, Subtractionem, Multiplicationem, Diviſionem, & Evectionem ad Poteſtates, quarum exponentes ſint numeri integri, non inſunt: erunt adeo $a + z$; $a - z$; az; $\frac{aa + zz}{a + z}$; $az^3 - bz^5$; &c. Functiones rationales ipſius z. At hujuſmodi expreſſiones $\sqrt{z}$; $a + \sqrt{(aa - zz)}$; $\sqrt[3]{(a - 2z + zz)}$: $\frac{aa - z\sqrt{(aa + zz)}}{a + z}$ erunt Functiones irrationales ipſius z.

Hæ commode diſtinguntur in Explicitas & Implicitas.

Explicitæ ſunt, quæ per ſigna radicalia ſunt evolutæ, cujuſmodi exempla modo ſunt data. Implicitæ vero Functiones irrationales ſunt quæ ex reſolutione æquationum ortum habent. Sic Z erit Functio irrationalis implicita ipſius z, ſi per hujuſmodi

æqua-

æquationem $Z^7 = azZ^2 - bz^5$ definiatur; quoniam valorem explicitum pro Z, admiſſis etiam ſignis radicalibus, exhibere non licet; propterea quod Algebra communis nondum ad hunc perfectionis gradum eſt evecta.

9. *Functiones rationales denuo ſubdividuntur in Integras & Fractas.*

In illis neque z uſquam habet exponentes negativos, neque expreſſiones continent fractiones, in quarum denominatores quantitas variabilis z ingrediatur: unde intelligitur Functiones fractas eſſe, in quibus denominatores z continentes, vel exponentes negativi ipſius z occurrant. Functionum integrarum hæc ergo erit Formula generalis: $a + bz + cz^2 + dz^3 + ez^4 + fz^5 +$ &c. nulla enim Functio ipſius z integra excogitari poteſt, quæ non in hac expreſſione contineatur. Functiones autem fractæ omnes, quia plures fractiones in unam cogi poſſunt, continebuntur in hac Formula:

$$\frac{a + bz + cz^2 + dz^3 + ez^4 + fz^5 + \&c.}{\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \zeta z^5 + \&c.}$$

ubi notandum eſt quantitates conſtantes a, b, c, d, &c. α, β, γ, δ, &c. ſive ſint affirmativæ, ſive negativæ, ſive integræ ſive fractæ, ſive rationales ſive irrationales, ſive etiam tranſcendentes, naturam Functionum non mutare.

10. *Deinde potiſſimum tenenda eſt Functionum diviſio in Uniformes ac Multiformes.*

Functio autem uniformis eſt, quæ ſi quantitati variabili z valor determinatus quicunque tribuatur, ipſa quoque unicum valorem determinatum obtineat. Functio autem Multiformis eſt, quæ, pro unoquoque valore determinato in locum variabilis z ſubſtituto, plures valores determinatos exhibet. Sunt igitur omnes Functiones rationales, ſive integræ ſive fractæ, Functiones uniformes; quoniam ejuſmodi expreſſiones, quicunque valor quantitati variabili tribuatur, non niſi unicum valorem præbent. Functiones autem irrationales omnes ſunt multiformes; propterea quod ſigna radicalia ſunt ambigua, & geminum valorem involvunt. Dantur autem quoque inter Functiones tranſcendentes,

LIB. I. tes, & uniformes, & multiformes: quin-etiam habentur Functiones infinitiformes; cujusmodi est Arcus Circuli Sinui z respondens; dantur enim Arcus circulares innumerabiles qui omnes eumdem habeant Sinum. Denotent autem hæ litteræ P, Q, R, S, T &c. singulæ Functiones uniformes ipsius z.

11. *Functio biformis ipsius* z *est ejusmodi Functio, quæ pro quovis ipsius* z *valore determinato, geminum valorem præbeat.*

Hujusmodi Functiones radices quadratæ exhibent, ut $\sqrt{(2z + zz)}$: quicunque enim valor pro z statuatur expressio $\sqrt{(2z+zz)}$ duplicem habet significatum, vel affirmativum vel negativum. Generatim vero Z erit Functio biformis ipsius z, si determinetur per æquationem quadraticam $Z^2 - PZ + Q = 0$: si quidem P & Q fuerint Functiones uniformes ipsius z. Erit namque $Z = \frac{1}{2}P \pm \sqrt{(\frac{1}{4}P^2 - Q)}$; ex quo patet cuique valori determinato ipsius z duplicem valorem determinatum ipsius Z respondere. Hic autem notandum est, vel utrumque valorem Functionis Z esse realem, vel utrumque imaginarium. Tum vero erit semper, uti constat ex natura æquationum, binorum valorum ipsius Z summa $= P$, ac productum $= Q$.

12. *Functio triformis ipsius* z *est, qua pro quovis ipsius* z *valore, tres valores determinatos exhibet.*

Hujusmodi Functiones ex resolutione æquationum cubicarum originem trahunt. Si enim fuerint P, Q, & R Functiones uniformes, sitque $Z^3 - PZ^2 + QZ - R = 0$, erit Z Functio triformis ipsius z; quia pro quolibet valore determinato ipsius z triplicem valorem obtinet. Tres isti ipsius Z valores unicuique valori ipsius z respondentes, vel erunt omnes reales, vel unicus erit realis, dum bini reliqui sunt imaginarii. Cæterum constat horum trium valorum summam perpetuo esse $= P$; summam factorum ex binis esse $= Q$, & productum ex omnibus tribus esse $= R$.

13. *Functio quadriformis ipsius* z *est, qua pro quovis ipsius* z *valore quatuor valores determinatos exhibet.*

Hujusmodi Functiones ex resolutione æquationum biquadraticarum

ticarum naſcuntur. Quod ſi enim P, Q, R, & S denotent Functiones uniformes ipſius z, fueritque $Z^4 - PZ^3 + QZ^2 - RZ + S = 0$, erit Z Functio quadriformis ipſius z; eo quod cuique valori ipſius z quadruplex valor ipſius Z reſpondet. Quatuor horum valorum ergo, vel omnes erunt reales, vel duo reales duoque imaginarii, vel omnes quatuor erunt imaginarii. Ceterum perpetuo ſumma horum quatuor valorum ipſius Z eſt $= P$, ſumma factorum ex binis $= Q$, ſumma factorum ex ternis $= R$, ac productum omnium $= S$. Simili autem modo comparata eſt ratio Functionum quinqueformium & ſequentium. CAP. I.

14. *Erit ergo Z Functio multiformis ipſius z, quæ, pro quovis valore ipſius z, tot exhibet valores quot numerus n continet unitates; ſi Z definiatur per hanc æquationem* $Z^n - PZ^{n-1} + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \&c. = 0$.

Ubi quidem notandum eſt n eſſe oportere numerum integrum; atque perpetuo, ut dijudicari poſſit quam multiformis ſit Functio Z ipſius z, æquatio, per quam Z definitur, reduci debet ad rationalitatem; quo facto exponens maximæ poteſtatis ipſius Z indicabit quæſitum valorum numerum cuique ipſius z valori reſpondentium. Deinde quoque tenendum eſt litteras P, Q, R, S, &c. denotare debere Functiones uniformes ipſius z: ſi enim aliqua earum jam eſſet Functio multiformis, tum Functio Z multo plures præbitura eſſet valores unicuique valori ipſius z reſpondentes, quam quidem numerus dimenſionum ipſius Z indicaret. Semper autem, ſi qui valores ipſius fuerint imaginarii, eorum numerus erit par; unde intelligitur, ſi fuerit n numerus impar, perpetuo unum ad minimum valorem ipſius Z fore realem: contra autem fieri poſſe, ſi numerus n fuerit par, ut nullus prorſus valor ipſius Z ſit realis.

15. *Si Z ejuſmodi fuerit Functio multiformis ipſius z ut perpetuo nonniſi unicum valorem exhibeat realem; tum Z Functionem uniformem ipſius z mentietur, ac pleramque loco Functionis uniformis uſurpari poterit.*

LIB. I. Ejusmodi Functiones erunt $\sqrt[3]{P}$, $\sqrt[5]{P}$, $\sqrt[7]{P}$, &c. quippe quæ perpetuo nonnisi unicum valorem realem præbent, reliquis omnibus existentibus imaginariis, dummodo P fuerit Functio uniformis ipsius z. Hanc ob rem hujusmodi expressio $P^{\frac{m}{n}}$, quoties n fuerit numerus impar, Functionibus uniformibus annumerari poterit; sive m fuerit numerus par sive impar. Quod si autem n fuerit numerus par, tum $P^{\frac{m}{n}}$ vel nullum habebit valorem realem, vel duos; ex quo ejusmodi expressiones $P^{\frac{m}{n}}$, existente n numero pari, eodem jure Functionibus biformibus accenseri poterunt: siquidem fractio $\frac{m}{n}$ ad minores terminos non fuerit reducibilis.

16. *Si fuerit* y *Functio quæcunque ipsius* z; *tum vicissim* z *erit Functio ipsius* y.

Cum enim y sit Functio ipsius z, sive uniformis sive multiformis; dabitur æquatio, qua y per z & constantes quantitates definitur. Ex eadem vero æquatione vicissim z per y & constantes definiri poterit; unde quoniam y est quantitas variabilis, z æquabitur expressioni ex y & constantibus compositæ, eritque adeo Functio ipsius y. Hinc quoque patebit quam multiformis Functio futura sit z ipsius y: fierique potest ut, etiamsi y fuerit Functio uniformis ipsius z, tamen z futura sit Functio multiformis ipsius y. Sic si y ex hac æquatione per z definiatur; $y^3 = ayz - bzz$; erit utique y Functio triformis ipsius z, contra vero z Functio tantum biformis ipsius y.

17. *Si fuerint* y *&* x *Functiones ipsius* z, *erit quoque* y *Functio ipsius* x, *& vicissim* x *Functio ipsius* y.

Cum enim sit y Functio ipsius z, erit quoque z Functio ipsius y: similique modo erit etiam z Functio ipsius x. Hanc ob rem Functio ipsius y æqualis erit Functioni ipsius x; ex qua æquatione & y per x & viceversa x per y definiri poterit: quocirca manifestum est esse y Functionem ipsius x, atque x Functionem ipsius

ipſius y. Sæpiſſime quidem has Functiones explicite exhibere non licet ob defectum Algebræ; interim tamen nihilo minus, quaſi omnes æquationes reſolvi poſſent, hæc Functionum reciprocatio perſpicitur. Ceterum per methodum in Algebra traditam, ex datis binis æquationibus, quarum altera continet y & z, altera vero x & z, per eliminationem quantitatis z formabitur una æquatio relationem inter x & y exprimens. CAP. I.

18 *Species denique quædam Functionum peculiares ſunt notandæ; ſic Functio par ipſius* z *eſt, quæ eundem dat valorem, ſive pro* z *ponatur valor determinatus* + k *ſive* — k.

Hujuſmodi ergo Functio par ipſius z erit zz; ſive enim ponatur $z = +k$, ſive $z = -k$, eundem valorem præbebit expreſſio zz, nempe $zz = +kk$. Simili modo Functiones pares ipſius z erunt hæ ipſius z poteſtates z^4, z^6, z^8, & generatim omnis poteſtas z^m, ſi fuerit m numerus par, ſive affirmativus ſive negativus. Quin etiam cum $z^{\frac{m}{n}}$ mentiatur Functionem ipſius z uniformem, ſi n ſit numerus impar, perſpicuum eſt $z^{\frac{m}{n}}$ fore Functionem parem ipſius z, ſi m fuerit numerus par, n vero numerus impar. Hanc ob rem, expreſſiones ex hujuſmodi poteſtatibus utcunque compoſitæ præbebunt Functiones pares ipſius z; ſic Z erit Functio par ipſius z, ſi fuerit $Z = a + bz^2 + cz^4 + dz^6 + \&c.$ item ſi fuerit $Z = \frac{a + bz^2 + cz^4 + dz^6 + \&c.}{\alpha + \beta z^2 + \gamma z^4 + \delta z^6 + \&c.}$; Similique modo exponentes fractos ipſius z introducendo, erit Z Functio par ipſius z ſi fuerit $Z = a + bz^{\frac{2}{3}} + cz^{\frac{2}{5}} + dz^{\frac{4}{7}}$ &c. vel $Z = a + bz^{-\frac{2}{3}} + cz^{-\frac{4}{3}} + dz^{-\frac{2}{5}} + \&c.$ vel $Z = \frac{a + bz^{\frac{2}{7}} + cz^{-\frac{4}{5}} + dz^{\frac{8}{3}}}{\alpha + \beta z^{\frac{2}{3}} + \gamma z^{-\frac{2}{5}} + \delta z^{\frac{4}{7}}}$. Cujuſmodi expreſſiones, cum omnes ſint Functiones uniformes ipſius z, appellari poterunt Functiones pares uniformes ipſius z.

LIB. I. 19. *Functio multiformis par ipsius z est, quæ etiam si pro quovis valore ipsius z plures exhibeat valores determinatos, tamen eosdem valores præbet, sive ponatur $z = +k$, sive $z = -k$.*

Sit Z ejusmodi Functio multiformis par ipsius z; quoniam natura Functionis multiformis exprimitur per æquationem inter Z & z, in qua Z tot habeat dimensiones, quot varios valores complectatur; manifestum est Z fore Functionem multiformem parem, si in æquatione naturam ipsius Z exprimente quantitas variabilis z ubique pares habeat dimensiones. Sic, si fuerit $Z^2 = azZ^4 + bz^2$, erit Z Functio biformis par ipsius z; sin autem sit $Z^3 - az^2Z^2 + bz^4Z - cz^8 = 0$, erit Z Functio triformis par ipsius z; atque generatim, si P, Q, R, S &c. denotent Functiones uniformes pares ipsius z, erit Z Functio biformis par ipsius z si sit $Z^2 - PZ + Q = 0$. At Z erit Functio triformis par ipsius z si sit $Z^3 - PZ^2 + QZ - R = 0$, & ita porro.

20 *Functio ergo, sive uniformis sive multiformis, par ipsius z erit ejusmodi expressio ex quantitate variabili z & constantibus conflata in qua ubique numerus dimensionum ipsius z sit par.*

Hujusmodi ergo Functiones, præter uniformes quarum exempla ante sunt allata, erunt hæ expressiones $a + \sqrt{(bb - zz)}$; $azz + \sqrt[3]{(a^6z^4 - bz^2)}$ item $az^{\frac{2}{3}} + \sqrt[3]{(z^2 + \sqrt{(a^4 - z^4)})}$ &c.

Unde patet Functiones pares ita definiri posse, ut dicantur esse Functiones ipsius zz.

Si enim ponatur $y = zz$, fueritque Z Functio quæcunque ipsius y; restituto ubique zz loco y, erit Z ejusmodi Functio ipsius z, in qua z ubique parem habeat dimensionum numerum. Excipiendi tamen sunt ii casus, quibus in expressione ipsius Z occurrunt $\sqrt{y}$: ac hujusmodi aliæ formæ, quæ, facto $y = zz$ signa radicalia amittunt. Quamvis enim sit $y + \sqrt{ay}$ Functio ipsius y, tamen posito $y = zz$, eadem expressio non erit Functio par ipsius z; cum fiat $y + \sqrt{ay} = zz + z\sqrt{a}$. Exclusis ergo his casibus, definitio ultima Functionum

num parium erit bona, atque ad ejusmodi Functiones formandas idonea. CAP. I.

21. *Functio impar ipsius z est ejusmodi Functio, cujus valor, si loco z ponatur — z, fit quoque negativus.*

Hujusmodi Functiones ergo impares erunt omnes potestates ipsius z, quarum exponentes sunt numeri impares, ut z^1, z^3, z^5, z^7; &c. item z^{-1}, z^{-3}, z^{-5}; &c. tum vero etiam $z^{\frac{m}{n}}$ erit Functio impar, si ambo numeri, m & n fuerint numeri impares. Generatim vero omnis expressio ex hujusmodi potestatibus composita erit Functio impar ipsius z; cujusmodi sunt, $az + bz^3 : az + az^{-1}$; item $z^{\frac{1}{3}} + az^{\frac{3}{5}} + bz^{-\frac{5}{3}}$; &c. Harum autem Functionum natura & inventio ex Functionibus paribus facilius perspicietur.

22. *Si Functio par ipsius z multiplicetur per z vel per ejusdem Functionem imparem quamcunque, productum erit Functio impar ipsius z.*

Sit P Functio par ipsius z, quæ idcirco manet eadem si loco z ponatur $-z$; quod si ergo in producto Pz, ponatur $-z$ loco z, prodibit $-Pz$; unde Pz erit Functio impar ipsius z. Sit jam P Functio par ipsius z, & Q functio impar ipsius z; atque ex Definitione patet si loco z ponatur $-z$, valorem ipsius P manere eundem, at valorem ipsius Q abire in sui negativum $-Q$; quare productum PQ, posito $-z$ loco z, abibit in $-PQ$, hoc est in sui negativum; eritque ideo PQ Functio impar ipsius z. Sic cum sit $a + \surd(aa + zz)$ functio par, & z^3 Functio impar ipsius z, erit productum $az^3 + z^3\surd(aa + zz)$ Functio impar ipsius z; similique modo $z \times \frac{a + bzz}{\alpha + \beta zz} = \frac{az + bz^3}{\alpha + \beta zz}$ Functio impar ipsius z. Ex his vero etiam intelligitur, si duarum Functionum P & Q, quarum altera P est par, altera Q, impar, altera per alteram dividatur, quotum fore Functionem imparem; erit ergo $\frac{P}{Q}$ itemque $\frac{Q}{P}$ Functio impar ipsius z.

 23. *Si*

23. *Si Functio impar per Functionem imparem vel multiplicetur, vel dividatur; quod resultat erit Functio par.*

Sint Q & S Functiones impares ipsius z; ita ut, posito $-z$ loco z, Q abeat in $-Q$, & S in $-S$; atque perspicuum est tam productum QS, quam quotum $\frac{Q}{S}$ eundem valorem retinere, etiamsi pro z ponatur $-z$; ideoque esse utrumque Functionem parem ipsius z. Manifestum itaque porro est cujusque Functionis imparis quadratum esse Functionem parem; cubum vero Functionem imparem; biquadratum iterum Functionem parem, atque ita porro.

24. *Si fuerit* y *Functio impar ipsius* z; *erit vicissim* z *Functio impar ipsius* y.

Cum enim sit y Functio impar ipsius z; si ponatur $-z$ loco z, abibit y in $-y$. Quod si ergo z per y definiatur, necesse est ut posito $-y$ loco y, quoque z abeat in $-z$; eritque ideo z Functio impar ipsius y. Sic quia, posito $y = z^3$, est y Functio impar ipsius z; erit quoque, ex æquatione $z^3 = y$ seu $z = y^{\frac{1}{3}}$, z Functio impar ipsius y. Et quia si fuerit $y = az + bz^3$, est y Functio impar ipsius z, erit vicissim, ex æquatione $bz^3 + az = y$, valor ipsius z per y expressus Functio impar ipsius y.

25. *Si natura Functionis* y *per ejusmodi æquationem definiatur, in cujus singulis terminis numerus dimensionum, quas* y *&* z *occupant conjunctim, sit vel par ubique, vel impar; tum erit* y *Functio impar ipsius* z.

Quod si enim in ejusmodi æquatione ubique loco z scribatur $-z$; simulque $-y$ loco y; omnes æquationis termini vel manebunt iidem, vel fient negativi, utroque vero casu æquatio manebit eadem. Unde patet $-y$ eodem modo per $-z$ determinatum iri, quo $+y$ per $+z$ determinatur; & hanc ob rem, si loco z ponatur $-z$, valor ipsius y abibit in $-y$, seu y erit Functio impar ipsius z. Sic si fuerit vel $yy = ayz + bzz + c$; vel $y^3 + ayyz = byzz + cy + dz$, ex utraque æquatione y erit Functio impar ipsius z.

26. *Si*

26. *Si Z fuerit Functio ipsius z, & Y Functio ipsius y, atque Y eodem modo definiatur per variabilem y & constantes, quo Z definitur per variabilem z & constantes; tum hæ Functiones Y et Z vocantur Functiones similes ipsarum y & z.* Cap. I.

Si scilicet fuerit $Z = a + bz + cz^2$, & $Y = a + by + cy^2$, erunt Z & Y Functiones similes ipsarum z & y, similique modo in multiformibus, si fuerit $Z^3 = azzZ + b$ & $Y^3 = ayyY + b$; erunt Z & Y Functiones similes ipsarum z & y. Hinc sequitur, si Y & Z fuerint hujusmodi Functiones similes ipsarum y & z; tum si loco z scribatur y, Functionem Z abituram esse in Functionem Y. Solet hæc similitudo etiam hoc modo verbis exprimi, ut Y talis Functio dicatur ipsius y, qualis Functio sit Z ipsius z. Hæ locutiones perinde occurrent, sive quantitates variabiles z & y a se invicem pendeant, sive secus: sic qualis Functio est $ay + by^3$ ipsius y, talis Functio erit $a(y+n) + b(y+n)^3$ ipsius $y+n$, existente scilicet $z = y + n$: tum qualis Functio est $\frac{a + bz + czz}{\alpha + \beta z + \gamma zz}$ ipsius z, talis Functio erit $\frac{azz + bz + c}{\alpha zz + \beta z + \gamma}$ ipsius $\frac{1}{z}$; posito $y = \frac{1}{z}$. Atque ex his luculenter perspicitur ratio similitudinis Functionum, cujus per universam Analysin sublimiorem uberrimus est usus. Ceterum hæc in genere de natura Functionum unius variabilis sufficere possunt; cum plenior expositio in applicatione sequente tradatur.

CAPUT II.

De transformatione Functionum.

27. *Functiones in alias formas transmutantur, vel loco quantitatis variabilis aliam introducendo, vel eandem quantitatem variabilem retinendo.*

Quod si eadem quantitas variabilis servatur, Functio proprie mutari non potest. Sed omnis transformatio consistit in alio

LIB. I. alio modo eandem Functionem exprimendi, quemadmodum ex Algebra constat eandem quantitatem per plures diversas formas exprimi posse. Hujusmodi transformationes sunt, si loco hujus Functionis $2 - 3z + zz$ ponatur $(1 - z)(2 - z)$, vel $(a + z)^3$ loco $a^3 + 3aaz + 3azz + z^3$, vel $\frac{a}{a - z} + \frac{a}{a + z}$ loco $\frac{2aa}{aa - zz}$; vel $\surd(1 + zz) + z$ loco $\frac{1}{\surd(1 + zz) - z}$; quæ expressiones, etsi forma differunt, tamen revera congruunt. Sæpe numero autem harum plurium formarum idem significantium una aptior est ad propositum efficiendum quam reliquæ, & hanc ob rem formam commodissimam eligi oportet.

Alter transformationis modus, quo loco quantitatis variabilis z alia quantitas variabilis y introducitur, quæ quidem ad z datam teneat relationem, per substitutionem fieri dicitur; hocque modo ita uti convenit, ut Functio proposita succinctius & commodius exprimatur, uti si ista proposita fuerit ipsius z Functio, $a^4 - 4a^3z + 6aazz - 4az^3 + z^4$; si loco $a - z$ ponatur y, prodibit ista multo simplicior ipsius y Functio y^4: &, si habeatur hæc Functio irrationalis $\surd(aa + zz)$ ipsius z, si ponatur $z = \frac{aa - yy}{2y}$, ista Functio per y expressa fiet rationalis $= \frac{aa + yy}{2y}$. Hunc autem transformationis modum in sequens Caput differam, hoc Capite illum, qui sine substitutione procedit, expositurus.

28. *Functio integra ipsius* z *sæpenumero commode in suos factores resolvitur, sicque in productum transformatur.*

Quando Functio integra hoc pacto in factores resolvitur, ejus natura multo facilius perspicitur; casus enim statim innotescunt, quibus Functionis valor fit $= 0$. Sic hæc ipsius z Functio $6 - 7z + z^3$ transformatur in hoc productum $(1 - z)(2 - z)(3 + z)$ ex quo statim liquet Functionem propositam tribus casibus fieri $= 0$; scilicet si $z = 1$, & $z = 2$, & $z = -3$, quæ proprietates ex forma $6 - 7z + z^3$ non tam facile intelliguntur. Istiusmodi Factores, in quibus variabiles z nulla

nulla occurrit poteſtas, vocantur Factores ſimplices, ut diſtinguantur a Factoribus compoſitis, in quibus ipſius z ineſt quadratum vel cubus, vel alia poteſtas altior. Erit ergo in genere $f+gz$ forma Factorum ſimplicium, $f+gz+hzz$ forma Factorum duplicium; $f+gz+hzz+iz^3$ formâ Factorum triplicium, & ita porro. Perſpicuum autem eſt Factorem duplicem duos complecti valores ſimplices, Factorem triplicem tres ſimplices, & ita porro. Hinc Functio ipſius z integra, in qua exponens ſummæ poteſtatis ipſius z eſt $=n$, continebit n Factores ſimplices; ex quo ſimul, ſi qui Factores fuerint vel duplices vel triplices, &c. numerus Factorum cognoſcetur. Cap. II.

29. *Factores ſimplices Functionis cujuſcunque integræ Z ipſius z reperiuntur, ſi Functio Z nihilo æqualis ponatur, atque ex hac æquatione omnes ipſius z radices inveſtigentur: ſingulæ enim ipſius z radices dabunt totidem Factores ſimplices Functionis Z.*

Quod ſi enim ex æquatione $Z=0$, fuerit quæpiam radix $z=f$, erit $z-f$ diviſor, ac proinde Factor Functionis Z, ſic igitur inveſtigandis omnibus radicibus æquationis $Z=0$, quæ ſint $z=f$, $z=g$, $z=h$; &c., Functio Z reſolvetur in ſuos Factores ſimplices, atque transformabitur in productum $Z=(z-f)(z-g)(z-h)$ &c.: ubi quidem notandum eſt ſi ſummæ poteſtatis ipſius z in Z non fuerit coefficiens $=+1$, tum productum $(z-f)(z-g)$ &c. inſuper per illum coefficientem multiplicari debere. Sic ſi fuerit $Z=Az^n+Bz^{n-1}+Cz^{n-2}+$ &c. erit $Z=A.(z-f).(z-g).(z-h).$ &c. At ſi fuerit $Z=A+Bz+Cz^2+Dz^3+Ez^4+$ &c. atque æquationis $Z=0$ radices z repertæ ſint; f; g; h; i; &c. erit $Z=A(1-\frac{z}{f})(1-\frac{z}{g})(1-\frac{z}{h})$. &c. Ex his autem viciſſim intelligitur, ſi Functionis Z Factor fuerit $z-f$, ſeu $1-\frac{z}{f}$; tum valorem Functionis in nihilum abire, ſi loco z ponatur f. Facto enim $z=f$, unus Factor $z-f$, ſeu $1-\frac{z}{f}$, Functionis Z, ideoque ipſa Functio Z evaneſcere debet.

30. *Factores simplices ergo erunt vel reales, vel imaginarii; &, si Functio Z habeat Factores imaginarios eorum nnmerus semper erit par.*

Cum enim Factores simplices nascantur ex radicibus æquationis $Z = 0$, radices reales præbebunt Factores reales, & imaginariæ imaginarios; in omni autem æquatione numerus radicum imaginariarum semper est par: quamobrem Functio Z, vel nullos habebit Factores imaginarios, vel duos, vel quatuor, vel sex, &c. Quod si Functio Z duos tantum habeat Factores imaginarios, eorum productum erit reale, ideoque præbebit Factorem duplicem realem. Sit enim $P =$ prodacto ex omnibus Factoribus realibus, erit productum duorum Factorum imaginariorum $= \frac{Z}{P}$; hincque reale. Simili modo si Functio Z habeat quatuor, vel sex, vel octo &c. Factores imaginarios; erit eorum productum semper reale: nempe æquale quoto, qui oritur, si Functio Z dividatur per productum omnium Factorum realium.

31. *Si fuerit Q productum reale ex quatuor Factoribus simplicibus imaginariis, tum idem hoc productum Q resolvi poterit in duos Factores duplices reales.*

Habebit enim Q ejusmodi formam $z^4 + Az^3 + Bz^2 + Cz + D$; quæ si negetur in duos Factores duplices reales resolvi posse, resolubilis erit statuenda in duos Factores duplices imaginarios; qui hujusmodi formam habebunt $zz - 2(p + q\sqrt{-1})z + r + s\sqrt{-1}$. & $zz - 2(p - q\sqrt{-1})z + r - s\sqrt{-1}$; aliæ enim formæ imaginariæ concipi non possunt, quarum productum fiat reale, nempe $= z^4 + Az^3 + Bz^2 + Cz + D$. Ex his autem Factoribus imaginariis duplicibus sequentes emergent quatuor Factores simplices imaginarii ipsius Q,

I. $z - (p + q\sqrt{-1}) + \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$
II. $z - (p + q\sqrt{-1}) - \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$
III. $z - (p - q\sqrt{-1}) + \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$
IV. $z - (p - q\sqrt{-1}) - \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$

Horum

Horum Factorum multiplicentur primus ac tertius in se invicem, posito brevitatis gratia, $t = pp - qq - r$, & $u = 2pq - s$; eritque horum Factorum productum $= zz - (2p - \sqrt{2t + 2\sqrt{(tt+uu)}})z$ Cap. II.

$$+ pp + qq - p\sqrt{2t + 2\sqrt{(tt+uu)}} + \sqrt{(tt+uu)};$$
$$+ q\sqrt{-2t + 2\sqrt{(tt+uu)}}$$

quod utique est reale. Simili autem modo productum ex Factoribus secundo & quarto erit reale nempe $= zz - (2p + \sqrt{2t + 2\sqrt{(tt+uu)}})z$

$$+ pp + qq + p\sqrt{2t + 2\sqrt{(tt+uu)}} + \sqrt{(tt+uu)}.$$
$$+ q\sqrt{-2t + 2\sqrt{(tt+uu)}}$$

Quocirca productum propositum Q, quod in duos Factores duplices reales resolvi posse negabatur, nihilo minus actu in duos Factores duplices reales est resolutum.

32. *Si Functio integra Z ipsius z quotcunque habeat Factores simplices imaginarios, bini semper ita conjungi possunt, ut eorum productum fiat reale.*

Quoniam numerus radicum imaginariarum semper est par, sit is $= 2n$; ac primo quidem patet productum harum radicum imaginariarum omnium esse reale. Quod si ergo duæ tantum radices imaginariæ habeantur, erit earum productum utique reale; sin autem quatuor habeantur Factores imaginarii, tum, uti vidimus, eorum productum resolvi potest in duos Factores duplices reales formæ $fzz + gz + h$. Quanquam autem eundem demonstrandi modum ad altiores potestates extendere non licet, tamen extra dubium videtur esse positum eandem proprietatem in quotcunque Factores imaginarios competere; ita ut semper loco $2n$ Factorum simplicium imaginariorum induci queant n Factores duplices reales. Hinc omnis Functio integra ipsius z resolvi poterit in Factores reales vel simplices vel duplices. Quod quamvis non summo rigore sit demonstratum, tamen ejus veritas in sequentibus magis corroborabitur, ubi hujus generis Functiones $a + bz^n$; $a + bz^n + cz^{2n}$; $a + bz^n + cz^{2n} + dz^{3n}$ &c. actu in istiusmodi Factores duplices reales resolventur.

LIB. I.

33. *Si Functio integra* Z, *posito* z = a *induat valorem* A, *&* *posito* z = b, *induat valorem* B; *tum*, *loco* z *valores medios inter* a *&* b *ponendo*, *Functio* Z *quosvis valores medios inter* A *&* B *accipere potest.*

Cum enim Z sit Functio uniformis ipsius z, quicunque valor realis ipsi z tribuatur, Functio quoque Z hinc valorem realem obtinebit. Cum igitur Z, priore casu $z = a$, nanciscatur valorem A; posteriore casu $z = b$, autem, valorem B; ab A ad B transire non poterit, nisi per omnes valores medios transeundo. Quod si ergo æquatio $Z - A = 0$ habeat radicem realem, simulque $Z - B = 0$ radicem realem suppeditet; tum æquatio quoque $Z - C = 0$ radicem habebit realem; si quidem C intra valores A & B contineatur. Hinc si expressiones $Z - A$ & $Z - B$ habeant Factorem simplicem realem, tum expressio quæcunque $Z - C$ Factorem simplicem habebit realem, dummodo C intra valores A & B contineatur.

34. *Si in Functione integra* Z *exponens maximæ ipsius* z *potestatis fuerit numerus impar* $2n+1$, *tum ea Functio* Z *unicum ad minimum habebit Factorem simplicem realem.*

Habebit scilicet Z hujusmodi formam $z^{2n+1} + \alpha z^{2n} + \beta z^{2n-1} + \gamma z^{2n-2} +$ &c. in qua si ponatur $z = \infty$, quia valores singulorum terminorum præ primo evanescunt, fiet $Z = (\infty)^{2n+1} = \infty$; ideoque $Z - \infty$ Factorem simplicem habebit realem nempe $z - \infty$. Sin autem ponatur $z = -\infty$, fiet $Z = (-\infty)^{2n+1} = -\infty$, ideoque habebit $Z + \infty$ Factorem simplicem realem $z + \infty$. Cum igitur tam $Z - \infty$, quam $Z + \infty$ habeat Factorem simplicem realem; sequitur etiam $Z + C$ habiturum esse Factorem simplicem realem, siquidem C contineatur intra limites $+\infty$ & $-\infty$; hoc est si C fuerit numerus realis quicunque, sive affirmativus, sive negativus. Hanc ob rem, facto $C = 0$, habebit quoque ipsa Functio Z Factorem simplicem realem $z - c$; atque quantitas c contine-

continebitur intra limites $+\infty$ & $-\infty$, eritque idcirco vel quantitas affirmativa, vel negativa, vel nihil. CAP. II.

35. *Functio igitur integra Z, in qua exponens maximæ potestatis ipsius z est numerus impar, vel unum habebit Factorem simplicem realem, vel tres, vel quinque, vel septem &c.*

Cum enim demonstratum sit Functionem Z certo unum habere Factorem simplicem realem $z-c$; ponamus eam præterea unum Factorem habere $z-d$, atque dividatur Functio Z, in qua maxima ipsius z potestas sit z^{2n+1}, per $(z-c).(z-d)$, erit quoti maxima potestas $= z^{2n-1}$, cujus exponens, cum sit numerus impar, indicat denuo ipsius Z dari Factorem simplicem realem. Si ergo Z plures uno habeat Factores simplices reales, habebit vel tres, vel (quoniam eodem modo progredi licet) quinque, vel septem, &c. Erit scilicet numerus Factorum simplicium realium impar, & quia numerus omnium Factorum simplicium est $= 2n+1$, erit numerus Factorum imaginariorum par.

36. *Functio integra Z, in qua exponens maximæ potestatis ipsius z est numerus par 2n, vel duos habebit Factores simplices reales vel quatuor, vel sex, vel &c.*

Ponamus ipsius Z constare Factorum simplicium realium numerum imparem $2m+1$; si ergo per horum omnium productum dividatur Functio Z, quoti maxima potestas erit $= z^{2n-2m-1}$, ejusque ideo exponens numerus impar; habebit ergo Functio Z præterea unum certo Factorem simplicem realem, ex quo numerus omnium Factorum simplicium realium ad minimum erit $= 2m+2$, ideoque par; ac numerus Factorum imaginariorum pariter par. Omnis ergo Functionis integræ Factores simplices imaginarii sunt numero pares; quemadmodum quidem jam ante statuimus.

37. *Si in Functione integra Z exponens maximæ potestatis ipsius z fuerit numerus par, atque terminus absolutus, seu constans, signo — affectus, tum Functio Z ad minimum duos habet Factores simplices reales.*

 Functio ergo Z, de qua hic ſermo eſt, hujuſmodi formam habebit $z^{2n} \pm \alpha z^{2n-1} \pm \beta z^{2n-2} \pm \ldots\ldots \pm \nu z - A$. Si jam ponatur $z = \infty$, fiet, uti ſupra vidimus, $Z = \infty$; atque, ſi ponatur $z = 0$, fiet $Z = -A$. Habebit ergo $Z - \infty$ Factorem realem $z - \infty$, & $Z + A$ Factorem $z - 0$: unde cum 0 contineatur intra limites $-\infty$ & $+A$, ſequitur $Z + 0$ habere Factorem ſimplicem realem $z - c$, ita ut c contineatur intra limites 0 & ∞. Deinde, cum poſito $z = -\infty$, fiat $Z = \infty$, ideoque $Z - \infty$ Factorem habeat $z + \infty$, & $Z + A$ Factorem $z + 0$, ſequitur quoque $Z + 0$ Factorem ſimplicem realem habere $z + d$; ita ut d intra limites 0 & ∞ contineatur; unde conſtat propoſitum. Ex his igitur perſpicitur ſi Z talis fuerit Functio, qualis hic eſt deſcripta, æquationem $Z = 0$, duas ad minimum habere debere radices reales, alteram affirmativam, alteram negativam. Sic æquatio hæc $z^4 + \alpha z^3 + \beta z^2 + \gamma z - aa = 0$, duas habet radices reales, alteram affirmativam, alteram negativam.

38. *Si in Functione fracta, quantitas variabilis* z *tot vel plures habeat dimenſiones in numeratore, quam in denominatore; tum iſta Functio reſolvi poterit in duas partes, quarum altera eſt Functio integra, altera fracta; in cujus numeratore quantitas variabilis* z *pauciores habeat dimenſiones quam in denominatore.*

Si enim exponens maximæ poteſtatis ipſius z minor fuerit in denominatore quam in numeratore; tum numerator per denominatorem dividatur more ſolito, donec in quoto ad exponentes negativos ipſius z perveniatur; hoc ergo loco abrupta diviſionis operatione quotus conſtabit ex parte integra atque fractione, in cujus numeratore minor erit dimenſionum numerus ipſius z quam in denominatore; hic autem quotus Functioni propoſitæ eſt æqualis. Sic, ſi hæc propoſita fuerit Functio fracta $\frac{1+z^4}{1+zz}$, ea per diviſionem ita reſolvetur.

$$\begin{array}{r|l} zz+1) & z^4+1\ \left(zz-1+\frac{2}{1+zz}\right. \\ & \underline{z^4+zz} \\ & -zz+1 \\ & \underline{-zz-1} \\ & +2 \end{array}$$

eritque $\frac{1+z^4}{1+zz} = zz - 1 + \frac{2}{1+zz}$. Hujusmodi Functiones fractæ, in quibus quantitas variabilis z tot vel plures habet dimensiones in numeratore quam in denominatore, ad similitudinem Arithmeticæ vocari possunt fractiones spuriæ, vel Functiones fractæ spuriæ, quo distinguantur a Functionibus fractis genuinis, in quarum numeratore quantitas variabilis z pauciores habet dimensiones quam in denominatore. Functio itaque fracta spuria resolvi poterit in Functionem integram, & Functionem fractam genuinam; hæcque resolutio per vulgarem divisionis operationem absolvetur.

39. *Si denominator Functionis fractæ duos habeat Factores inter se primos; tum ipsa Functio fracta resolvetur in duas fractiones, quarum denominatores sint illis binis Factoribus respective æquales.*

Quanquam hæc resolutio ad Functiones fractas spurias æque pertinet atque ad genuinas, tamen eam ad genuinas potissimum accomodabimus. Resoluto autem denominatore hujusmodi Functionis fractæ in duos Factores inter se primos, ipsa Functio resolvetur in duas alias Functiones fractas genuinas, quarum denominatores sint illis binis Factoribus respective æquales; hæcque resolutio, si quidem fractiones sint genuinæ, unico modo fieri potest; cujus rei veritas ex exemplo clarius quam per ratiocinium perspicietur. Sit ergo proposita hæc Functio fracta $\frac{1-2z+3zz-4z^3}{1+4z^4}$, cujus denominator $1+4z^4$ cum sit æqualis huic producto $(1+2z+2zz)(1-2z+2zz)$, fractio proposita in duas fractiones resolvetur, quarum alterius denominator erit $1+2z+2zz$, alterius $1-2z+2zz$: ad quas inveniendas, quia sunt genuinæ, statuantur numeratores illius $= \alpha + 6z$, hujus $= \gamma + \delta z$, eritque per hypothesin

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LIB. I. $\frac{1 - 2z + 3zz - 4z^3}{1 + 4z^4} = \frac{\alpha + \beta z}{1 + 2z + 2zz} + \frac{\gamma + \delta z}{1 - 2z + 2zz}$: addantur actu hæ duæ fractiones, eritque summæ

Numerator	Denominator
$+\alpha - 2\alpha z + 2\alpha zz$ $+\beta z - 2\beta zz + 2\beta z^3$ $+\gamma + 2\gamma z + 2\gamma zz$ $+\delta z + 2\delta zz + 2\delta z^3$	$1 + 4z^4$

Cum ergo denominator æqualis sit denominatori fractionis propositæ, numeratores quoque æquales reddi debent: quod, ob tot litteras incognitas $\alpha, \beta, \gamma, \delta$, quot sunt termini æquales efficiendi, utique fieri, idque unico modo poterit: nanciscimur scilicet has quatuor æquationes

I. $\alpha + \gamma = 1$

II. $-2\alpha + \beta + 2\gamma + \delta = -2$

III. $2\alpha - 2\beta + 2\gamma + 2\delta = 3$

IV. $2\beta + 2\delta = -4$.

Hinc ob $\alpha + \gamma = 1$, & $\beta + \delta = -2$; æquationes II. & III. dabunt $\alpha - \gamma = 0$ & $\delta - \beta = \frac{1}{2}$; ex quibus fit $\alpha = \frac{1}{2}$; $\gamma = \frac{1}{2}$; $\beta = \frac{-5}{4}$; $\delta = \frac{-3}{4}$; ideoque fractio proposita $\frac{1 - 2z + 3zz - 4z^3}{1 + 4z^4}$, transformatur in has duas $\frac{\frac{1}{2} - \frac{5}{4}z}{1 + 2z + 2zz} + \frac{\frac{1}{2} - \frac{3}{4}z}{1 - 2z + 2zz}$. Simili autem modo facile perspicietur resolutionem semper succedere debere: quoniam semper tot litteræ incognitæ introducuntur, quot opus est ad numeratorem propositum eliciendum. Ex doctrina vero fractionum communi intelligitur hanc resolutionem succedere non posse, nisi isti denominatoris Factores fuerint inter se primi.

40. *Functio igitur fracta* $\frac{M}{N}$ *in tot fractiones simplices formæ* $\frac{A}{p - qz}$ *resolvi poterit, quot Factores simplices habet denominator* N *inter se inæquales.*

Repræ-

Repræſentat hic fractio $\frac{M}{N}$ Functionem quamcunque fractam genuinam, ita ut M & N ſint Functiones integræ ipſius z, atque ſumma poteſtas ipſius z in M minor ſit quam in N. Quod ſi ergo denominator N in ſuos Factores ſimplices reſolvatur, hique inter ſe fuerint inæquales, expreſſio $\frac{M}{N}$ in tot fractiones reſolvetur, quot Factores ſimplices in denominatore N continentur; propterea quod quiſque Factor abit in denominatorem fractionis partialis. Si ergo $p - qz$ fuerit Factor ipſius N, is erit denominator fractionis cujuſdam partialis, &, cum in numeratore hujus fractionis numerus dimenſionum ipſius z minor eſſe debeat quam in denominatore $p - qz$, numerator neceſſario erit quantitas conſtans. Hinc ex unoquoque Factore ſimplici $p - qz$ denominatoris N naſcetur fractio ſimplex $\frac{A}{p - qz}$; ita ut ſumma omnium harum fractionum ſit æqualis fractioni propoſitæ $\frac{M}{N}$. Cap. II.

EXEMPLUM.

Sit, exempli cauſa, propoſita hæc Functio fracta $\frac{1 + zz}{z - z^3}$; quia Factores ſimplices denominatoris ſunt z, $1 - z$, & $1 + z$, iſta Functio reſolvetur in has tres fractiones ſimplices $\frac{A}{z} + \frac{B}{1 - z} + \frac{C}{1 + z} = \frac{1 + zz}{z - z^3}$; ubi numeratores conſtantes A, B, & C definire oportet. Reducantur hæ fractiones ad communem denominatorem, qui erit $z - z^3$; atque numeratorum ſumma æquari debebit ipſi $1 + zz$, unde iſta æquatio oritur:

$$\begin{array}{l} A + Bz - Azz = 1 + zz = 1 + 0z + zz. \\ \quad + Cz + Bzz \\ \qquad\quad - Czz \end{array}$$

LIB. I. quæ totidem comparationes præbet, quot ſunt litteræ incognitæ A, B, C; erit ſcilicet,

I°. $A = 1$.
II°. $B + C = 0$.
III°. $- A + B - C = 1$:

Hinc fit $B - C = 2$; & porro $A = 1$; $B = 1$ & $C = -1$. Functio ergo propoſita $\frac{1 + zz}{z - z^3}$ reſolvitur in hanc formam $\frac{1}{z} + \frac{1}{1 - z} - \frac{1}{1 + z}$. Simili autem modo intelligitur, quotcunque habuerit denominator N Factores ſimplices inter ſe inæquales, ſemper fractionem $\frac{M}{N}$ in totidem fractiones ſimplices reſolvi. Sin autem aliquot Factores fuerint æquales inter ſe, tum alio modo poſt-explicando reſolutio inſtitui debet.

41. *Cum igitur quilibet Factor ſimplex denominatoris* N *ſuppeditet fractionem ſimplicem pro reſolutione Functionis propoſitæ* $\frac{M}{N}$; *oſtendendum eſt quomodo ex Factore ſimplice denominatoris* N *cognito, fractio ſimplex reſpondens reperiatur.*

Sit $p - qz$ Factor ſimplex ipſius N, ita ut ſit $N = (p - qz)S$; atque S Functio integra ipſius z; ponatur fractio ex Factore $p - qz$ orta $= \frac{A}{p - qz}$, & ſit fractio ex altero Factore denominatoris S oriunda $= \frac{P}{S}$, ita ut, ſecundum §. 39., futurum ſit $\frac{M}{N} = \frac{A}{p - qz} + \frac{P}{S} = \frac{M}{(p - qz)S}$; hinc erit $\frac{P}{S} = \frac{M - AS}{(p - qz)S}$; quæ fractiones cum congruere debeant, neceſſe eſt ut $M - AS$ ſit diviſibile per $p - qz$; quoniam Functio integra P ipſi quoto æquatur. Quando vero $p - qz$ Diviſor exiſtit ipſius $M - AS$, hæc expreſſio poſito $z = \frac{p}{q}$ evaneſcit. Ponatur ergo ubique loco z hic valor conſtans $\frac{p}{q}$ in M

&

& S, erit $M - AS = 0$, ex quo fiet $A = \frac{M}{S}$; hocque ergo modo reperitur numerator A fractionis quæsitæ $\frac{A}{p - qz}$; atque si ex singulis denominatoris N Factoribus simplicibus, dummodo sint inter se inæquales, hujusmodi fractiones simplices formentur, harum fractionum simplicium omnium summa erit æqualis Functioni propositæ $\frac{M}{N}$. CAP. II.

EXEMPLUM.

Sic, si in Exemplo præcedente $\frac{1 + zz}{z - z^3}$, ubi est $M = 1 + zz$, & $N = z - z^3$, sumatur z pro Factore simplice, erit $S = 1 - zz$, atque fractionis simplicis $\frac{A}{z}$ hinc ortæ erit numerator $A = \frac{1 + zz}{1 - zz} = 1$ posito $z = 0$, quem valorem z obtinet si ipse hic Factor simplex z nihilo æqualis ponatur. Simili modo si pro denominatoris Factore sumatur $1 - z$, ut sit $S = z + zz$ erit $A = \frac{1 + zz}{z + zz}$, facto $1 - z = 0$, unde erit $A = 1$, & ex Factore $1 - z$ nascitur fractio $\frac{1}{1 - z}$. Tertius denique Factor $1 + z$, ob $S = z - zz$, & $A = \frac{1 + zz}{z - zz}$, posito $1 + z = 0$, seu $z = -1$, dabit $A = -1$, & fractionem simplicem $= \frac{-1}{1 + z}$. Quare per hanc regulam reperitur $\frac{1 + zz}{z - z^3} = \frac{1}{z} + \frac{1}{1 - z} - \frac{1}{1 + z}$, ut ante.

42. *Functio fracta hujus formæ* $\frac{P}{(p - qz)^n}$, *cujus numerator* P *non tantam ipsius* z *potestatem involvit quantam denominator* $(p - qz)^n$, *transmutari potest in hujusmodi fractiones partiales*

LIB. I. $\frac{A}{(p-qz)^n} + \frac{B}{(p-qz)^{n-1}} + \frac{C}{(p-qz)^{n-2}} + \ldots + \frac{K}{p-qz}$; *quarum omnium numeratores ſint quantitates conſtantes.*

Quoniam maxima poteſtas ipſius z in P minor eſt quam z^n, erit z^{n-1}, ideoque P hujuſmodi habebit formam:

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \ldots\ldots\ldots + \varkappa z^{n-1}$$

exiſtente terminorum numero $= n$, cui æquari debet numerator ſummæ omnium fractionum partialium, poſtquam ſingulæ ad eundem denominatorem $(p-qz)^n$ fuerint perductæ: qui numerator propterea erit $= A + B(p-qz) + C(p-qz)^2 + D(p-qz)^3 + \ldots\ldots + K(p-qz)^{n-1}$. Hujus maxima ipſius z poteſtas eſt, ut ibi, z^{n-1}, atque tot habentur litteræ incognitæ A, B, C, ... K, (quarum numerus eſt $= n$,) quot ſunt termini congruentes reddendi. Quamobrem litteræ conſtantes A, B, C, &c. ita definiri poterunt, ut fiat Functio fracta genuina $\frac{P}{(p-qz)^n} = \frac{A}{(p-qz)^n} + \frac{B}{(p-qz)^{n-1}} + \frac{C}{(p-qz)^{n-2}} + \frac{D}{(p-qz)^{n-3}} + \ldots\ldots + \frac{K}{p-qz}$. Ipſa autem horum numeratorum inventio mox facilis aperietur.

43. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p-qz)^2$, *ſequenti modo fractiones partiales ex hoc Factore oriundæ reperientur.*

Cujuſmodi fractiones partiales ex ſingulis Factoribus denominatoris ſimplicibus, qui alios ſibi æquales non habeant, oriantur, ante eſt oſtenſum: nunc igitur ponamus duos Factores inter ſe eſſe æquales, ſeu, iis conjunctis, denominatoris N Factorem eſſe $(p-qz)^2$. Ex hoc ergo Factore per §. præced. duæ naſcentur fractiones partiales hæ $\frac{A}{(p-qz)^2} + \frac{B}{p-qz}$. Sit autem

tem $N = (p - qz)^2 S$, eritque $\frac{M}{N} = \frac{M}{(p-qz)^2 S} = \frac{A}{(p-qz)^2}$ CAP. II.
$+ \frac{B}{p-qz} + \frac{P}{S}$, denotante $\frac{P}{S}$ omnes fractiones simplices junctim sumptas ex denominatoris Factore S ortas. Hinc erit $\frac{P}{S}$
$= \frac{M - AS - B(p-qz)S}{(p-qz)^2 S}$, & $P = \frac{M - AS - B(p-qz)S}{(p-qz)^2}$
$=$ Functioni integræ. Debet ergo $M - AS - B(p-qz)S$ divisibile esse per $(p-qz)^2$: sit primum divisibile per $p-qz$, atque tota expressio $M - AS - B(p-qz)S$ evanescet, posito $p - qz = 0$, seu $z = \frac{p}{q}$; ponatur ergo ubique $\frac{p}{q}$ loco z, eritque $M - AS = 0$, ideoque $A = \frac{M}{S}$; scilicet fractio $\frac{M}{S}$, si loco z ubique ponatur $\frac{p}{q}$, dabit valorem ipsius A constantem. Hoc invento quantitas $M - AS - B(p-qz)S$ etiam per $(p-qz)^2$ divisibilis esse debet, seu $\frac{M-AS}{p-qz} -$ BS denuo per $p-qz$ divisibile esse debet. Posito ergo ubique $z = \frac{p}{q}$ erit $\frac{M-AS}{p-qz} = BS$, ideoque $B = \frac{M-AS}{(p-qz)S}$ $= \frac{1}{p-qz}\left(\frac{M}{S} - A\right)$, ubi notandum est, cum $M - AS$ divisibile sit per $p-qz$, hanc divisionem prius institui debere, quam loco z substituatur $\frac{p}{q}$. Vel ponatur $\frac{M-AS}{p-qz} = T$, eritque $B = \frac{T}{S}$ posito $z = \frac{p}{q}$; inventis ergo numeratoribus A & B, erunt fractiones partiales ex denominatoris N Factore $(p-qz)^2$ ortæ hæ $\frac{A}{(p-qz)^2} + \frac{B}{p-qz}$.

EXEMPLUM I.

Sit hæc proposita Functio fracta $\frac{1-zz}{zz(1+zz)}$ erit, ob de-

 nomina-

LIB. I. nominatoris Factorem quadratum zz; $S = 1 + zz$ & $M = 1 - zz$. Sint fractiones partiales ex zz ortæ $\frac{A}{zz} + \frac{B}{z}$, erit $A = \frac{M}{S} = \frac{1 - zz}{1 + zz}$, posito Factore $z = 0$; hincque $A = 1$. Tum erit $M - AS = - 2zz$ quod divisum per Factorem simplicem z, dabit $T = - 2z$, hincque $B = \frac{T}{S} = \frac{-2z}{1 + zz}$, posito $z = 0$; unde erit $B = 0$; atque ex Factore denominatoris zz orietur unica hæc fractio partialis $\frac{1}{zz}$.

EXEMPLUM II.

Sit hæc proposita Functio fracta $\frac{z^3}{(1-z)^2(1+z^4)}$, cujus, ob denominatoris Factorem quadratum $(1-z)^2$, fractiones partiales sint $\frac{A}{(1-z)^2} + \frac{B}{1-z}$. Erit ergo $M = z^3$ & $S = 1 + z^4$; ideoque $A = \frac{M}{S} = \frac{z^3}{1+z^4}$, posito $1 - z = 0$, seu $z = 1$: unde fit $A = \frac{1}{2}$. Prodibit ergo $M - AS = z^3 - \frac{1}{2} - \frac{1}{2}z^4 = - \frac{1}{2} + z^3 - \frac{1}{2}z^4$, quod per $1 - z$ divisum dat $T = - \frac{1}{2} - \frac{1}{2}z - \frac{1}{2}zz + \frac{1}{2}z^3$; ideoque $B = \frac{T}{S} = \frac{-1-z-zz+z^3}{2+2z^4}$, posito $z = 1$; ita ut sit $B = -\frac{1}{2}$; fractiones ergo partiales quæsitæ sunt $\frac{1}{2(1-z)^2} - \frac{1}{2(1-z)}$.

44. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p - qz)^3$ *sequenti modo fractiones partiales ex hoc Factore oriundæ* $\frac{A}{(p-qz)^3} + \frac{B}{(p-qz)^2} + \frac{C}{p-qz}$ *reperientur.*

Ponatur

Ponatur $N = (p - qz)^3 S$, sitque fractio ex Factore S orta $= \frac{P}{S}$, erit $P = \frac{M - AS - B(p-qz)S - C(p-qz)^2 S}{(p-qz)^3}$ = Functioni integræ. Numerator ergo $M - AS - B(p-qz)S - C(p-qz)^2 S$ ante omnia divisibilis esse debet per $(p-qz)$; unde is, posito $p - qz = 0$, seu $z = \frac{p}{q}$, evanescere debet, eritque adeo $M - AS = 0$, ideoque $A = \frac{M}{S}$, posito $z = \frac{p}{q}$. Invento hoc pacto A erit $M - AS$ divisibile per $p - qz$ ponatur ergo $\frac{M - AS}{p - qz} = T$, atque $T - BS - C(p-qz)S$ adhuc per $(p-qz)^2$ erit divisibile; fiet ergo $= 0$, posito $p - qz = 0$; ex quo prodit $B = \frac{T}{S}$ posito $z = \frac{p}{q}$. Sic autem invento B erit $T - BS$ divisibile per $p - qz$. Hanc ob rem, posito $\frac{T - BS}{p - qz} = V$, superest ut $V - CS$ divisibile sit per $p - qz$; eritque ergo $V - CS = 0$, posito $p - qz = 0$, atque $C = \frac{V}{S}$, posito $z = \frac{p}{q}$. Inventis ergo hoc modo numeratoribus A, B, C, fractiones partiales ex denominatoris N Factore $(p-qz)^3$ ortæ erunt $\frac{A}{(p-qz)^3} + \frac{B}{(p-qz)^2} + \frac{C}{p-qz}$.

CAP. II.

EXEMPLUM.

Sit proposita hæc fracta Functio $\frac{zz}{(1-z)^3(1+zz)}$, ex cujus denominatoris Factore cubico $(1-z)^3$ oriantur hæ fractiones partiales: $\frac{A}{(1-z)^3} + \frac{B}{(1-z)^2} + \frac{C}{1-z}$. Erit ergo $M = zz$ & $S = 1 + zz$; unde fit primum $A = \frac{zz}{1+zz}$

posito

LIB. I. posito $1 - z = 0$ seu $z = 1$; ex quo prodit $A = \frac{1}{2}$. Jam ponatur $T = \frac{M - AS}{1 - z}$, erit $T = \frac{\frac{1}{2}zz - \frac{1}{2}}{1 - z} = -\frac{1}{2} - \frac{1}{2}z$; unde oritur $B = \frac{-\frac{1}{2} - \frac{1}{2}z}{1 + zz}$, posito $z = 1$, ita ut sit $B = -\frac{1}{2}$. Ponatur porro $V = \frac{T - BS}{1 - z} = \frac{T + \frac{1}{2}S}{1 - z}$; erit $V = \frac{-\frac{1}{2}z + \frac{1}{2}zz}{1 - z} = -\frac{1}{2}z$; unde fit $C = \frac{V}{S} = \frac{-\frac{1}{2}z}{1 + zz}$ posito $z = 1$, ita ut sit $C = -\frac{1}{4}$. Quo circa fractiones partiales ex denominatoris Factore $(1 - z)^3$ ortæ erunt $\frac{1}{2(1 - z)^3} - \frac{1}{2(1 - z)^2} - \frac{1}{4(1 - z)}$.

45. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p - qz)^n$; *fractiones partiales hinc ortæ* $\frac{A}{(p - qz)^n} + \frac{B}{(p - qz)^{n-1}} + \frac{C}{(p - qz)^{n-2}} + \dots\dots + \frac{K}{p - qz}$ *sequenti modo invenientur.*

Ponatur denominator $N = (p - qz)^n Z$, atque, ratiocinium ut ante instituendo, reperietur ut sequitur:

Primo $A = \frac{M}{Z}$, posito $z = \frac{p}{q}$. Ponatur $P = \frac{M - AZ}{p - qz}$

Secundo $B = \frac{P}{Z}$, posito $z = \frac{p}{q}$. Ponatur $Q = \frac{P - BZ}{p - qz}$

Tertio $C = \frac{Q}{Z}$, posito $z = \frac{p}{q}$. Ponatur $R = \frac{Q - CZ}{p - qz}$

Quarto $D = \frac{R}{Z}$, posito $z = \frac{p}{q}$. Ponatur $S = \frac{R - DZ}{p - qz}$

Quinto $E = \frac{S}{Z}$, posito $z = \frac{p}{q}$. &c.

Hoc ergo modo si definiantur singuli numeratores constantes

tes *A*, *B*, *C*, *D*, &c. invenientur omnes fractiones partiales, quæ ex denominatoris *N* Factore $(p - qz)^n$ nascuntur. CAP. II.

EXEMPLUM.

Sit proposita ista Functio fracta $\frac{1+zz}{z^5(1+z^3)}$ ex cujus denominatoris Factore z^5 nascantur hæ fractiones partiales $\frac{A}{z^5} + \frac{B}{z^4} + \frac{C}{z^3} + \frac{D}{z^2} + \frac{E}{z}$. Ad quarum numeratores constantes inveniendos, erit $M = 1 + zz$ atque $Z = 1 + z^3$; & $\frac{p}{q} = 0$. Sequens ergo calculus ineatur.

Primum est $A = \frac{M}{Z} = \frac{1+zz}{1+z^3}$, posito $z = 0$; ergo $A = 1$.

Ponatur $P = \frac{M - AZ}{z} = \frac{zz - z^3}{z} = z - zz$. Eritque secundo $B = \frac{P}{Z} = \frac{z - zz}{1+z^3}$, posito $z = 0$; ergo $B = 0$.

Ponatur $Q = \frac{P - BZ}{z} = \frac{z - zz}{z} = 1 - z$; eritque tertio $C = \frac{Q}{Z} = \frac{1-z}{1+z^3}$, posito $z = 0$: ergo $C = 1$.

Ponatur $R = \frac{Q - CZ}{z} = \frac{-z - z^3}{z} = -1 - zz$; erit quarto $D = \frac{R}{Z} = \frac{-1-zz}{1+z^5}$, posito $z = 0$; ex quo fit $D = -1$.

Ponatur $S = \frac{R - DZ}{z} = \frac{-zz + z^3}{z} = -z + zz$; erit quinto $E = \frac{S}{Z} = \frac{-z+zz}{1+z^3}$, posito $z = 0$; unde fit $E = 0$.

Quocirca fractiones partiales quæsitæ erunt hæ:

$\frac{1}{z^5} + \frac{0}{z^4} + \frac{1}{z^3} - \frac{1}{z^2} + \frac{0}{z}$.

LIB. I.

46. *Quæcunque ergo proposita fuerit Functio rationalis fracta $\frac{M}{N}$, ea sequenti modo in partes resolvetur, atque in formam simplicissimam transmutabitur.*

Quærantur denominatoris N omnes Factores simplices sive reales sive imaginarii; quorum qui sibi pares non habeant, seorsim tractentur & ex unoquoque per §. 41, fractio partialis eruatur. Quod si idem Factor simplex bis vel pluries occurrat, ii conjunctim sumantur atque ex eorum producto, quod erit potestas formæ $(p - qz)^n$, quærantur fractiones partiales convenientes per §. 45. Hocque modo cum ex singulis Factoribus simplicibus denominatoris erutæ fuerint fractiones partiales, tum harum omnium aggregatum æquabitur Functioni propositæ $\frac{M}{N}$, nisi fuerit spuria; si enim fuerit spuria, pars integra insuper extrahi atque ad istas fractiones partiales inventas adjici debebit, quo prodeat valor Functionis $\frac{M}{N}$ in forma simplicissima expressus. Perinde autem est sive fractiones partiales ante extractionem partis integræ, sive post quærantur. Eædem enim ex singulis denominatoris N Factoribus prodeunt fractiones partiales, sive adhibeatur ipse numerator M, sive idem quocunque denominatoris N multiplo vel auctus vel minutus; id quod regulas datas contemplanti facile patebit.

EXEMPLUM.

Quæratur valor Functionis $\frac{1}{z^3(1-z)^2(1+z)}$ in forma simplicissima expressus. Sumatur primum Factor denominatoris solitarius $1+z$, qui dat $\frac{p}{q} = -1$. erit $M = 1$ & $Z = z^3 - 2z^4 + z^5$. Hinc ad fractionem $\frac{A}{1+z}$ inveniendam, erit $A = \frac{1}{z^3 - 2z^4 + z^5}$, posito $z = -1$; ideoque fit $A =$

—

$-\frac{1}{4}$, atque ex Factore $1+z$ oritur hæc fractio partialis Cap. II.
$\frac{-1}{4(1+z)}$. Jam ſumatur Factor quadratus $(1-z)^2$ qui dat $\frac{p}{q}=1$. $M=1$, & $Z=z^3+z^4$; poſitis ergo fractionibus partialibus hinc ortis $\frac{A}{(1-z)^2}+\frac{B}{1-z}$, erit $A=\frac{1}{z^3+z^4}$, poſito $z=1$; ergo $A=\frac{1}{2}$; fiat $P=\frac{M-\frac{1}{2}Z}{1-z}$ $=\frac{1-\frac{1}{2}z^3-\frac{1}{2}z^4}{1-z}=1+z+zz+\frac{1}{2}z^3$; eritque $B=\frac{P}{Z}=\frac{1+z+zz+\frac{1}{2}z^3}{z^3+z^4}$, poſito $z=1$; ergo $B=\frac{7}{4}$ & fractiones partiales quæſitæ $\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}$. Denique tertius Factor cubicus z^3 dat $\frac{p}{q}=0$; $M=1$; & $Z=1-z-zz+z^3$. Poſitis ergo fractionibus partialibus his $\frac{A}{z^3}+\frac{B}{z^2}+\frac{C}{z}$; erit primum. $A=\frac{M}{N}=\frac{1}{1-z-zz+z^3}$ poſito $z=0$; ergo $A=1$. Ponatur $P=\frac{M-Z}{z}=1+z-zz$, erit $B=\frac{P}{Z}$, poſito $z=0$; ergo $B=1$. Ponatur $Q=\frac{P-Z}{z}=2-zz$; erit $C=\frac{Q}{Z}$, poſito $z=0$; ergo $C=2$. Hanc ob rem Functio propoſita $\frac{1}{z^3(1-z)^2(1+z)}$ in hanc formam reſolvitur $\frac{1}{z^3}+\frac{1}{z^2}+\frac{2}{z}+\frac{1}{2(1-z)^2}$ $+\frac{7}{4(1-z)}-\frac{1}{4(1+z)}$: nulla enim pars integra inſuper accedit, quia fractio propoſita non eſt ſpuria.

CAPUT III.

De transformatione Functionum per substitutionem.

46. *Si fuerit* y *Functio quæcunque ipsius* z, *atque* z *definiatur per novam variabilem* x, *tum quoque* y *per* x *definiri poterit.*

Cum ergo antea y fuisset Functio ipsius z, nunc nova quantitas variabilis x inducitur, per quam utraque priorum y & z definiatur. Sic, si fuerit $y = \frac{1 - zz}{1 + zz}$, atque ponatur $z = \frac{1 - x}{1 + x}$; hoc valore loco z substituto, erit $y = \frac{2x}{1 + xx}$. Sumpto ergo pro x valore quocunque determinato, ex eo reperientur valores determinati pro z & y, sicque invenitur valor ipsius y respondens illi valori ipsius z qui simul prodiit. Uti si sit $x = \frac{1}{2}$, fiet $z = \frac{1}{3}$, & $y = \frac{4}{5}$; reperitur autem quoque $y = \frac{4}{5}$, si in $\frac{1 - zz}{1 + zz}$, cui expressioni y æquatur, ponatur $z = \frac{1}{3}$.

Adhibetur autem hæc novæ variabilis introductio ad duplicem finem: vel enim hoc modo irrationalitas, qua expressio ipsius y per z data laborat, tollitur; vel quando ob æquationem altioris gradus, qua relatio inter y & z exprimitur, non licet Functionem explicitam ipsius z ipsi y æqualem exhibere, nova variabilis x introducitur, ex qua utraque y & z commode definiri queat: unde insignis substitutionum usus jam satis elucet, ex sequentibus vero multo clarius perspicietur.

47. *Si fuerit* $y = \sqrt{(a + bz)}$; *nova variabilis* x *per quam utraque* z & y *rationaliter exprimatur, sequenti modo invenietur.*

Quoniam tam z quam y debet esse Functio rationalis ipsius x; perspicuum est hoc obtineri si ponatur $\sqrt{(a + bz)} = bx$: Fiet enim primo $y = bx$; & $a + bz = bbxx$; hincque $z = bxx - \frac{a}{b}$. Quocirca utraque quantitas y & z per Functionem rationalem ipsius x exprimitur; scilicet cum sit $y = \sqrt{(a + bz)}$ fiat $z = bxx - \frac{a}{b}$; erit $y = bx$.

48. S.

48. *Si fuerit* $y = (a+bz)^{m:n}$; *nova variabilis* x, *per quam tam* y *quam* z *rationaliter exprimatur*, *ſic reperietur.*

Ponatur $y = x^m$, fietque $(a+bz)^{m:n} = x^m$ ideoque $(a+bz)^{1:n} = x$: ergo $a+bz = x^n$ & $z = \frac{x^n - a}{b}$. Sic ergo utraque quantitas y & z rationaliter per x definietur, ope ſcilicet ſubſtitutionis $z = \frac{x^n - a}{b}$, quæ præbet $y = x^m$. Quamvis igitur neque y per z, neque viciſſim z per y rationaliter exprimi poſſit; tamen utraque reddita eſt Functio rationalis novæ quantitatis variabilis x per ſubſtitutionem introductæ, ſcopo ſubſtitutionis omnino convenienter.

49. *Si fuerit* $y = \left(\frac{a+bz}{f+gz}\right)^{m:n}$; *requiritur nova quantitas variabilis* x *per quam utraque* y & z *rationaliter exprimatur.*

Manifeſtum primo eſt ſi ponatur $y = x^m$, quæſito ſatisfieri; erit enim $\left(\frac{a+bz}{f+gz}\right)^{m:n} = x^m$, ideoque $\frac{a+bz}{f+gz} = x^n$; ex qua æquatione elicitur $z = \frac{a - fx^n}{gx^n - b}$; quæ ſubſtitutio præbet $y = x^m$.

Hinc quoque intelligitur ſi fuerit $\left(\frac{\alpha+\beta y}{\gamma+\delta y}\right)^n = \left(\frac{a+bz}{f+gz}\right)^m$, tam y quam z rationaliter per x expreſſum iri, ſi utraque formula ponatur $= x^{mn}$; reperietur enim $y = \frac{\alpha - \gamma x^m}{\delta x^m - \beta}$ & $z = \frac{a - fx^n}{gx^n - b}$: qui caſus nil habent difficultatis.

50. *Si fuerit* $y = \sqrt{((a+bz)(c+dz))}$; *ſubſtitutio idonea invenietur*, *qua* y & z *rationaliter exprimuntur*, *hoc modo.*

Ponatur $\sqrt{((a+bz)(c+dz))} = (a+bz)x$, facile enim perſpicitur hinc valorem rationalem pro z eſſe proditurum; quia valor ipſius z per æquationem ſimplicem determinatur. Erit

 ergo

 ergo $c + dz = (a + bz) xx$, hincque $z = \frac{c - axx}{bxx - d}$. Quare porro fiet $a + bz = \frac{bc - ad}{bxx - d}$; & ob $y = \surd((a+bz)(c+dz)) = (a+bz)x$ habebitur $y = \frac{(bc - ad)x}{bxx - d}$. Functio ergo irrationalis $y = \surd((a+bz)(c+dz))$ ad rationalitatem perducitur ope substitutionis $z = \frac{c - axx}{bxx - d}$, quippe quæ dabit $y = \frac{(bc - ad)x}{bxx - d}$. Sic, si fuerit $y = \surd(aa - zz) = \surd((a+z)(a-z))$; ob $b = +1$; $c = a$, $d = -1$, ponatur $z = \frac{a - axx}{1 + xx}$, eritque $y = \frac{2ax}{1 + xx}$. Quoties ergo quantitas post signum $\surd$ habuerit duos Factores simplices reales, hoc modo reductio ad rationalitatem absolvetur; sin autem Factores bini simplices fuerint imaginarii, sequenti modo uti præstabit.

51. *Sit* $y = \surd(p + qz + rzz)$; *atque requiritur substitutio idonea pro* z *facienda, ut valor ipsius* y *fiat rationalis.*

Pluribus modis hoc fieri potest, prout p & q fuerint quantitates vel affirmativæ vel negativæ. Sit primo p quantitas affirmativa, ac ponatur aa pro p; etiamsi enim p non sit quadratum, tamen irrationalitas quantitatum constantium præsens negotium non turbat. Sit igitur

I. $y = \surd(aa + bz + czz)$; ac ponatur $\surd(aa + bz + czz) = a + xz$; erit $b + cz = 2ax + xxz$; unde fit $z = \frac{b - 2ax}{xx - c}$: tum vero erit $y = a + xz = \frac{bx - axx - ac}{xx - c}$; ubi z & y sunt Functiones rationales ipsius x. Sit jam

II. $y = \surd(aazz + bz + c)$; ac ponatur $\surd(aazz + bz + c) = az + x$; erit $bz + c = 2axz + xx$, & $z = \frac{xx - c}{b - 2ax}$. Tum autem fit $y = az + x = \frac{-ac + bx - axx}{b - 2ax}$.

III. Si fuerint p & r quantitates negativæ; tum, nisi sit $qq > 4pr$, valor ipsius y semper erit imaginarius. Quod si autem fuerit $qq > 4pr$; expressio $p + qz + rzz$ in duos Factores resolvi

resolvi poterit, qui casus ad §. praeced. reducitur. Saepenumero autem commodius ad hanc formam reducitur, $y = \sqrt{(aa + (b+cz)(d+ez))}$; pro qua ad rationalitatem perducenda ponatur $y = a + (b+cz)x$, eritque $d + ez = 2ax + bxx + cxxz$; unde fit $z = \frac{d - 2ax - bxx}{cxx - e}$, & $y = \frac{-ae + (cd - be)x - acxx}{cxx - e}$. Interdum commodius fieri potest reductio ad hanc formam, $y = \sqrt{(aazz + (b+cz)(d+ez))}$. Tum ponatur $y = az + (b+cz)x$; erit $d + ez = 2axz + bxx + cxxz$ & $z = \frac{bxx - d}{e - 2ax - cxx}$, atque $y = \frac{-ad + (be - cd)x - abxx}{e - 2ax - cxx}$.

EXEMPLUM.

Si habeatur ista ipsius z Functio irrationalis $y = \sqrt{(-1 + 3z - zz)}$; quae cum reduci queat ad hanc formam $y = \sqrt{(1 - 2 + 3z - zz)} = \sqrt{(1 - (1-z)(2-z))}$; ponatur $y = 1 - (1-z)x$, erit $-2 + z = -2x + xx - xxz$ & $z = \frac{2 - 2x + xx}{1 + xx}$. Deinde est $1 - z = \frac{-1 + 2x}{1 + xx}$ & $y = 1 - (1-z)x = \frac{1 + x - xx}{1 + xx}$. Atque hi sunt fere casus, quos Algebra indeterminata, seu methodus *Diophantaea*, suppeditat; neque alios casus in his tractatis non comprehensos per substitutionem rationalem ad rationalitatem reducere licet. Quocirca ad alterum substitutionis usum monstrandum progredior.

52. *Si* y *ejusmodi fuerit Functio ipsius* z *ut sit* $ay^{\alpha} + bz^{\beta} + cy^{\gamma} z^{\delta} = 0$, *invenire novam variabilem* x, *per quam valores ipsarum* y *&* z *explicite assignari queant.*

Quoniam resolutio aequationum generalis non habetur, ex aequatione proposita $ay^{\alpha} + bz^{\beta} + cy^{\gamma} z^{\delta} = 0$ neque y per

z neque

LIB. I. z neque vicissim z per y exhiberi potest. Quo igitur huic incommodo remedium afferatur; ponatur $y = x^m z^n$, eritque $ax^{\alpha m} z^{\alpha n} + bz^{\beta} + cx^{\gamma m} z^{\gamma n + \delta} = 0$. Determinetur nunc exponens n ita ut ex hac æquatione valor ipsius z definiri queat: quod tribus modis præstari potest.

I. Sit $\alpha n = \beta$; ideoque $n = \frac{\beta}{\alpha}$; erit, æquatione per $z^{\alpha n} = z^{\beta}$ divisa, $ax^{\alpha m} + b + cx^{\gamma m} z^{\gamma n - \beta + \delta} = 0$; unde oritur

$$z = \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{1}{\gamma n - \beta + \delta}}, \text{ sive}$$

$$z = \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{\alpha}{\beta\gamma - \alpha\beta + \alpha\delta}}, \ \&$$

$$y = x^m \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{\beta}{\beta\gamma - \alpha\beta + \alpha\delta}}$$

II. Sit $\beta = \gamma n + \delta$ seu $n = \frac{\beta - \delta}{\gamma}$; erit, æquatione per z^{β} divisa, $ax^{\alpha m} z^{\alpha n - \beta} + b + cx^{\gamma m} = 0$; unde oritur

$$z = \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{1}{\alpha n - \beta}} = \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\gamma}{\alpha\beta - \alpha\delta - \beta\gamma}},$$

atque $$y = x^m \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\beta - \delta}{\alpha\beta - \alpha\delta - \beta\gamma}}.$$

III. Sit $\alpha n = \gamma n + \delta$, seu $n = \frac{\delta}{\alpha - \gamma}$; erit, æquatione per $z^{\alpha n}$ divisa, $ax^{\alpha m} + bz^{\beta - \alpha n} + cx^{\gamma m} = 0$; unde oritur

$$z = \left(\frac{-ax^{\alpha m} - cx^{\gamma m}}{b}\right)^{\frac{1}{\beta - \alpha n}} =$$

$$\left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{\alpha-\gamma}{\alpha\beta-\beta\gamma-\alpha\delta}}; \text{ atque}$$

$$y=x^m\left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{\delta}{\alpha\beta-\beta\gamma-\alpha\delta}}.$$

Tribus igitur diverſis modis erutæ ſunt Functiones ipſius x, quæ ipſis z & y ſunt æquales. Præterea vero pro m numerum pro lubitu ſubſtituere licet cyphra excepta; ſicque formulæ ad commodiſſimam expreſſionem reduci poterunt.

EXEMPLUM.

Exprimatur natura Functionis y per hanc æquationem $y^3+z^3-cyz=0$; atque quærantur Functiones ipſius x ipſis y & z æquales. Erit ergo $a=-1$; $b=-1$; $\alpha=3$; $\beta=3$; $\gamma=1$; & $\delta=1$. Hinc primus modus dabit, poſito $m=1$, $z=\left(\frac{x^3+1}{cx}\right)^{-1}$ & $y=x\left(\frac{x^3+1}{cx}\right)^{-1}$, ſive $z=\frac{cx}{1+x^3}$ & $y=\frac{cxx}{1+x^3}$; quarum expreſſionum utraque adeo eſt rationalis.

Secundus modus vero dabit hos valores:

$$z=\left(\frac{cx-1}{x^3}\right)^{1:3}, \& \ y=x\left(\frac{cx-1}{x^3}\right)^{2:3}, \text{ ſive}$$

$$z=\frac{1}{x}\sqrt[3]{(cx-1)}, \& \ y=\frac{1}{x}\sqrt[3]{(cx-1)^2}.$$

Tertius modus ita rem expediet ut ſit

$$z=(cx-x^3)^{2:3}, \& \ y=x(cx-x^3)^{1:3}.$$

53. *Hinc a poſteriori intelligitur cujuſmodi æquationes, quibus valor Functionis* y *per* z *determinatur, hoc modo novam variabilem* x *introducendo reſolvi queant.*

Ponamus enim reſolutione jam inſtituta prodiiſſe has deter-

Lib. I

minationes $z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{p:r}$, atque $y = x$ $\left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{q:r}$; eritque $y^p = x^p z^q$; & hinc $x = yz^{-q:p}$. Cum igitur sit $z^{r:p} = \frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}$, si loco x ejus valorem $y z^{-q:p}$ substituamus; prodibit ista æquatio $z^{r:p} = \frac{ay^\alpha z^{-\alpha q:p} + by^\beta z^{-\beta q:p} + cy^\gamma z^{-\gamma q:p} + \&c.}{A + By^\mu z^{-\mu q:p} + Cy^\nu z^{-\nu q:p} + \&c.}$; quæ reducitur ad hanc $Az^{r:p} + By^\mu z^{(r - \mu q):p} + Cy^\nu z^{(r - \nu q):q} + \&c. = ay^\alpha z^{-\alpha q:p} + by^\beta z^{-\beta q:p} + cy^\gamma z^{-\gamma q:p} + \&c.$ quæ multiplicata per $z^{\alpha q:p}$ transibit in hanc: $Az^{(\alpha q + r):p} + By^\mu z^{(\alpha q - \mu q + r):p} + Cy^\nu z^{(\alpha q - \nu q + r):p} + \&c. = ay^\alpha + by^\beta z^{(\alpha q - \beta q):p} + cy^\gamma z^{(\alpha q - \gamma q):p} + \&c.$ Ponatur $\frac{\alpha q + r}{p} = m$ & $\frac{\alpha q - \beta q}{p} = n$: fiet $p = \alpha - \beta$; $q = n$, & $r = \alpha m - \beta m - \alpha n$: atque nascetur ista æquatio: $Az^m + By^\mu z^{m - \mu n:(\alpha - \beta)} + Cy^\nu z^{m - \nu n:(\alpha - \beta)} + \&c. = ay^\alpha + by^\beta z^n + cy^\gamma z^{(\alpha - \gamma)n:(\alpha - \beta)} + \&c.$ quæ propterea ita resolvetur ut sit:

$$z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{\alpha - \beta}{\alpha m - \beta m - \alpha n}} \;\&$$

$$y = x\left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{n}{\alpha m - \beta m - \alpha n}}.$$

Vel ponatur $\frac{\alpha q + r}{p} = m$, & $\frac{\alpha q - \mu q + r}{p} = n$, erit $m - n$

=

$= \frac{\mu q}{p}$; & $\frac{q}{p} = \frac{m - n}{\mu}$, atque $\frac{r}{p} = m - \frac{\alpha m + \alpha n}{\mu}$. Hinc fit $p = \mu$; $q = m - n$; & $r = \mu m - \alpha m + \alpha n$; atque hæc æquatio resultabit: CAP. III.

$$Az^m + By^\mu z^n + Cy^\nu z^{\mu m - \nu(m-n):\mu} + \&c. = ay^\alpha + by^\beta z^{(\alpha-\beta)(m-n):\mu} + cy^\gamma z^{(\alpha-\gamma)(m-n):\mu}$$

+ &c. quæ ita resolvetur ut sit:

$$z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{\mu}{\mu m - \alpha m + \alpha n}} \&$$

$$y = x\left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{m-n}{\mu m - \alpha m + \alpha n}}$$

54. *Si* y *ita pendeat a* z *ut sit* ayy + byz + czz + dy + ez = o, *sequenti modo tam* y *quam* z *rationaliter per novam variabilem* x *exprimetur.*

Ponatur $y = xz$, erit divisione per z facta: $axxz + bxz + cz + dx + e = o$, ex qua reperitur $z = \frac{-dx - e}{axx + bx + c}$, & $y = \frac{-dxx - ex}{axx + bx + c}$.

At vero ad formam propositam reduci potest hæc æquatio inter y & z: $ayy + byz + czz + dy + ez + f = o$ diminuendo vel augendo utramque variabilem certa quadam quantitate constante, unde & hæc æquatio per novam variabilem x rationaliter explicari potest.

55. *Si* y *ita pendeat a* z, *ut sit* $ay^3 + by^2z + cyz^2 + dz^3 + eyy + fyz + gzz = o$; *sequenti modo tam* y *quam* z *rationaliter per novam variabilem* x *exprimi poterit.*

Ponatur $y = xz$, & facta substitutione tota æquatio per zz dividi poterit: prodibit autem $ax^3z + bxxz + cxz + dz + exx + fx + g = o$. Unde oritur $z = \frac{-exx - fx - g}{ax^3 + bxx + cx + d}$ ex quo erit $y = \frac{-ex^3 - fxx - gx}{ax^3 + bxx + cx + d}$.

LIB. I. Ex his casibus facile intelligitur quemadmodum æquationes altiorum graduum, quibus y per z definitur, comparatæ esse debeant, ut hujusmodi resolutio locum habere queat. Ceterum hi casus in superioribus formulis §. 53. continentur: at, quia formulæ generales non tam facile ad hujusmodi casus sæpius occurrentes accommodantur, visum est horum aliquos seorsim evolvere.

56. *Si* y *ita pendeat a* z *ut sit* $ayy + byz + czz = d$ *hoc modo utraque quantitas* y *&* z *per novam variabilem* x *exprimetur.*

Ponatur $y = xz$, eritque $(axx + bx + c)zz = d$, ideoque $z = \sqrt{\frac{d}{axx + bx + c}}$ & $y = x\sqrt{\frac{d}{axx + bx + c}}$.

Simili modo si fuerit, $ay^3 + by^2z + cyz^2 + dz^3 = ey + fz$; posito $y = xz$, tota æquatio per z divisa dabit $(ax^3 + bxx + cx + d)zz = ex + f$; unde oritur $z = \sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$; & $y = x\sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$. Hi autem casus aliique similes resolutiones admittentes comprehenduntur in sequente paragrapho.

57. *Si* y *ita pendeat a* z *ut sit* $ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \&c. = \alpha y^n + \beta y^{n-1}z + \gamma y^{n-2}z^2 + \delta y^{n-3}z^3 + \&c.$ *Sequenti modo tam* z *quam* y *commode per novam variabilem* x *exprimetur.*

Sit $y = xz$, atque facta substitutione tota æquatio dividi poterit per z^n, siquidem exponens m sit major quam n; eritque $(ax^m + bx^{m-1} + cx^{m-2} + \&c.)z^{m-n} = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \&c.$ unde obtinebitur

$z =$

$$z = \left(\frac{\alpha x^{n} + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \&c.}{a x^{m} + b x^{m-1} + c x^{m-2} + d x^{m-3} + \&c.} \right)^{1:(m-n)} \quad \&$$

$$y = x \left(\frac{\alpha x^{n} + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \&c.}{a x^{m} + b x^{m-1} + c x^{m-2} + d x^{m-3} + \&c.} \right)^{1:(m-n)}$$

Hæc ſcilicet reſolutio locum habet, ſi in æquatione naturam Functionis y per z exprimente, duplex tantum ubique occurrit dimenſionum ab y & z ſumptarum numerus; uti in caſu tractato in ſingulis terminis numerus dimenſionum vel eſt m vel n.

58. *Si in æquatione inter* y *&* z *triplicis generis dimenſiones occurrant, quarum ſumma tantum ſuperet mediam, quantum hæc media infimam, ope reſolutionis æquationis quadratæ variabiles* y *&* z *per novam* x *exprimi poterunt.*

Si enim ponatur $y = xz$, diviſione per minimam ipſius z poteſtatem facta, valor ipſius z per x, ope extractionis radicis quadratæ exhibebitur, id quod ex ſequentibus exemplis erit manifeſtum.

EXEMPLUM I.

Sit $ay^3 + by^2z + cyzz + dz^3 = 2eyy + 2fyz + 2gzz + hy + iz$; ac ponatur $y = xz$: erit, diviſione per z facta, $(ax^3 + bxx + cx + d)zz = 2(exx + fx + g)z + hx + i$; ex qua ſequens ipſius z obtinebitur valor:

$$z = \frac{exx + fx + g \pm \sqrt{((exx + fx + g)^2 + (ax^3 + bxx + cx + d)(hx + i))}}{ax^3 + bxx + cx + d}$$

quo invento erit $y = xz$.

EXEMPLUM II.

Sit $y^5 = 2az^3 + by + cz$; ac, poſito $y = xz$, erit $x^5z^4 = 2azz + bx + c$; ex qua reperitur $zz = \frac{a \pm \sqrt{(aa + bx^6 + cx^5)}}{x^5}$; & $z = \frac{\sqrt{(a \pm \sqrt{(aa + bx^6 + cx^5)})}}{xx\sqrt{x}}$ & $y = \frac{\sqrt{(a \pm \sqrt{(aa + bx^6 + cx^5)})}}{x\sqrt{x}}$.

EXEMPLUM III.

Sit $y^{10} = 2ayz^6 + byz^3 + cz^4$, in qua cum dimenſiones ſint 10, 7, & 4, ponatur $y = xz$; atque æquatio per z^4 diviſa abibit in hanc: $x^{10} z^6 = 2axz^3 + bx + c$ ſeu $z^6 = \frac{2axz^3 + bx + c}{x^{10}}$; unde invenitur $z^3 = \frac{ax \pm x\sqrt{(aa + bx^9 + cx^8)}}{x^{10}}$; ideoque erit $z = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^3}$; atque $y = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^2}$. Ex quibus exemplis uſus hujuſmodi ſubſtitutionum abunde perſpicitur.

CAPUT IV.

De explicatione Functionum per ſeries infinitas.

59. CUm Functiones fractæ atque irrationales ipſius z non in forma integra $A + Bz + Cz^2 + Dz^3 +$ &c. continentur, ita ut terminorum numerus ſit finitus, quæri ſolent hujuſmodi expreſſiones in infinitum excurrentes, quæ valorem cujuſvis Functionis ſive fractæ ſive irrationalis exhibeant. Quin etiam natura Functionum tranſcendentium melius intelligi cenſetur, ſi per ejuſmodi formam, etſi infinitam, exprimantur. Cum enim natura Functionis integræ optime perſpiciatur, ſi ſecundum diverſas poteſtates ipſius z explicetur, atque adeo ad formam $A + Bz + Cz^2 + Dz^3 +$ &c. reducatur, ita eadem forma aptiſſima videtur ad reliquarum Functionum omnium indolem menti repræſentandam, etiamſi terminorum numerus ſit revera infinitus. Perſpicuum autem eſt nullam Functionem non integram ipſius z per numerum hujuſmodi terminorum $A + Bz + Cz^2 +$ &c. finitum exponi poſſe; eo ipſo enim Functio

Functio foret integra; num vero per hujusmodi terminorum seriem infinitam exhiberi possit, si quis dubitet, hoc dubium per ipsam evolutionem cujusque Functionis tolletur. Quo autem hæc explicatio latius pateat, præter potestates ipsius z exponentes integros affirmativos habentes, admitti debent potestates quæcunque. Sic dubium erit nullum quin omnis Functio ipsius z in hujusmodi expressionem infinitam transmutari possit: $Az^{\alpha} + Bz^{\beta} + Cz^{\gamma} + Dz^{\delta} +$ &c. denotantibus exponentibus α, β, γ, δ, &c. numeros quoscunque.

60. *Per divisionem autem continuam intelligitur fractionem* $\frac{a}{\alpha + \beta z}$ *resolvi in hanc seriem infinitam* $\frac{a}{\alpha} - \frac{a\beta z}{\alpha^2} + \frac{a\beta^2 z^2}{\alpha^3} - \frac{a\beta^3 z^3}{\alpha^4} + \frac{a\beta^4 z^4}{\alpha^5} -$ *&c., quæ, cum quilibet terminus ad sequentem habeat rationem constantem* $1 : \frac{\beta z}{\alpha}$, *vocatur series geometrica.*

Potest vero quoque hæc series ita inveniri, ut ipsa initio pro incognita habeatur: ponatur enim $\frac{a}{\alpha + \beta z} = A + Bz + Cz^2 + Dz^3 + Ez^4 +$ &c. atque ad æqualitatem producendam quærantur coëfficientes A, B, C, D, &c. Erit ergo $a = (\alpha + \beta z)(A + Bz + Cz^2 + Dz^3 +$ &c.$)$, & multiplicatione actu peracta fiet

$$\begin{aligned} a = \alpha A &+ \alpha Bz + \alpha Cz^2 + \alpha Dz^3 + \alpha Ez^4 + \&c. \\ &+ \beta Az + \beta Bz^2 + \beta Cz^3 + \beta Dz^4 + \&c. \end{aligned}$$

Quamobrem esse debet $a = \alpha A$, ideoque $A = \frac{a}{\alpha}$, & coëfficientium uniuscujusque potestatis ipsius z summa nihilo æqualis est ponenda: unde prodibunt hæ æquationes,

$\alpha B + \beta A = 0$	cognito ergo quovis coëfficiente
$\alpha C + \beta B = 0$	facile reperitur sequens; si enim
$\alpha D + \beta C = 0$	fuerit coëfficiens termini cujusque $= P$
$\alpha E + \beta D = 0$	& sequens $= Q$ erit $\alpha Q + \beta P = 0$
&c.	sive $Q = \frac{\beta P}{\alpha}$.

Cum

LIB. I. Cum igitur terminus primus A ſit determinatus $= \frac{a}{\alpha}$ ex eo ſequentes litteræ B, C, D, &c. definiuntur eodem modo, quo ex diviſione ſunt orti. Ceterum ex inſpectione perſpicitur in ſerie infinita pro $\frac{a}{\alpha + \beta z}$ inventa poteſtatis z^n coëfficientem fore $= \pm \frac{a \beta^n}{\alpha^{n+1}}$, ubi ſignum $+$ locum habet ſi n ſit numerus par, ſignum $-$ autem ſi n ſit numerus impar: ſeu coëfficiens erit $= \frac{a}{\alpha}\left(\frac{-\beta}{\alpha}\right)^n$.

61. *Simili modo ope diviſionis continuatæ hæc Functio fracta* $\frac{a + bz}{\alpha + \beta z + \gamma zz}$ *in ſeriem infinitam converti poteſt.*

Cum autem diviſio ſit tædioſa, neque tam facile naturam ſeriei infinitæ oſtendat, commodius erit ſeriem quæſitam fingere, atque modo ante tradito determinare. Sit igitur

$$\frac{a + bz}{\alpha + \beta z + \gamma zz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$$

multiplicetur utrinque per $\alpha + \beta z + \gamma zz$, atque fiet

$$\begin{aligned} a + bz = \alpha A + \alpha Bz &+ \alpha Cz^2 + \alpha Dz^3 + \alpha Ez^4 + \&c. \\ + \beta Az &+ \beta Bz^2 + \beta Cz^3 + \beta Dz^4 + \&c. \\ &+ \gamma Az^2 + \gamma Bz^3 + \gamma Cz^4 + \&c. \end{aligned}$$

Hinc erit $\alpha A = a$; $\alpha B + \beta A = b$; unde reperitur $A = \frac{a}{\alpha}$ & $B = \frac{b}{\alpha} - \frac{a\beta}{\alpha\alpha}$; reliquæ vero litteræ ex ſequentibus æquationibus determinabuntur:

$\alpha C + \beta B + \gamma A = 0$
$\alpha D + \beta C + \gamma B = 0$
$\alpha E + \beta D + \gamma C = 0$
$\alpha F + \beta E + \gamma D = 0$
&c.

hinc ergo ex binis quibuſque coëfficientibus contiguis ſequens reperitur. Sic ſi duo coëfficientes contigui fuerint P, Q & ſequens R, erit $\alpha R + \beta Q + \gamma P = 0$ ſeu $R = \frac{-\beta Q - \gamma P}{\alpha}$.

Cum igitur duæ litteræ primæ A & B jam ſint inventæ ſequentes C, D, E, F &c. omnes ſucceſſive ex iis invenientur,

tur, ficque reperietur Series infinita $A + Bz + Cz^2 + Dz^3 +$ &c. Cap. IV.
Functioni fractæ propofitæ $\frac{a + bz}{\alpha + \beta z + \gamma z^2}$ æqualis.

EXEMPLUM.

Si fuerit propofita hæc fractio $\frac{1 + 2z}{1 - z - zz}$, huicque æqualis ftatuatur Series $A + Bz + Cz^2 + Dz^3 +$ &c. ob $a = 1$; $b = 2$; $\alpha = 1$; $\beta = -1$; $\gamma = -1$; erit $A = 1$; $B = 3$; tum vero erit

$C = B + A$
$D = C + B$
$E = D + C$
$F = E + D$
&c.

quilibet ergo coëfficiens æqualis eft fummæ duorum præcedentium; quare fi cogniti fuerint duo coëfficientes contigui P & Q, erit fequens $R = P + Q$.

Cum igitur duo coëfficientes primi A & B fint cogniti, fractio propofita $\frac{1 + 2z}{1 - z - zz}$ in hanc Seriem infinitam tranfmutatur $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 +$ &c., quæ nullo negotio quoufque libuerit continuari poteft.

62. Ex his jam fatis intelligitur indoles Serierum infinitarum, in quas Functiones fractæ tranfmutantur; tenent enim ejufmodi legem, ut quilibet terminus ex aliquot præcedentibus determinari poffit. Scilicet, fi denominator fractionis propofitæ fuerit $\alpha + \beta z$, atque Series infinita ftatuatur

$$A + Bz + Cz^2 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + \&c.;$$

quilibet coëfficiens Q ex præcedente P folo ita definietur ut fit $\alpha Q + \beta P = 0$. Sin denominator fuerit trinomium $\alpha + \beta z + \gamma zz$, quilibet coëfficiens Seriei R ex duobus præcedentibus Q & P ita definietur ut fit $\alpha R + \beta Q + \gamma P = 0$: fimili modo fi denominator fuerit quadrinomium, ut $\alpha + \beta z + \gamma zz + \delta z^3$, quilibet coëfficiens feriei S ex tribus antecedentibus R, Q & P ita determinabitur, ut fit $\alpha S + \beta R + \gamma Q + \delta P = 0$,

 ficque

 ſicque de ceteris. In his ergo Seriebus quilibet terminus determinatur ex aliquot antecedentibus ſecundum legem quandam conſtantem, quæ lex ex denominatore fractionis hanc Seriem producentis ſponte apparet. Vocari autem hæ Series a Celeb. MOIVRÆO, qui earum naturam maxime eſt ſcrutatus, ſolent *recurrentes*, propterea quod ad terminos antecedentes eſt recurrendum, ſi ſequentes inveſtigare velimus.

63. Ad harum porro Serierum formationem requiritur ut denominatoris terminus conſtans α non ſit $= 0$: cum enim inventus ſit terminus Seriei primus $A = \frac{a}{\alpha}$, tum is, tum omnes ſequentes fierent infiniti, ſi eſſet $\alpha = 0$. Hoc ergo caſu excluſo, quem deinceps evolvam, Functio fracta in Seriem infinitam recurrentem tranſmutanda, hujuſmodi habebit formam $\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$; ubi primum denominatoris terminum pono $= 1$, huc enim ſemper fractio reduci poteſt, niſi is ſit $= 0$; reliquos autem denominatoris terminos omnes tanquam negativos contemplor, ut Seriei hinc formatæ omnes termini fiant affirmativi. Quod ſi enim Series recurrens hinc orta ponatur $A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$ coëfficientes ita determinabuntur ut ſit

$$\begin{aligned}
A &= a \\
B &= \alpha A + b \\
C &= \alpha B + \beta A + c \\
D &= \alpha C + \beta B + \gamma A + d \\
E &= \alpha D + \beta C + \gamma B + \delta A + e \\
&\&c.
\end{aligned}$$

Quilibet ergo coëfficiens æqualis eſt aggregato ex multiplis aliquot præcedentium una cum numero quodam, quem numerator præbet. Niſi autem numerator in infinitum progrediatur, hæc additio mox ceſſabit, atque quivis terminus ſecundum legem conſtantem ex aliquot præcedentibus determinabitur. Ne ergo lex progreſſionis uſquam turbetur conveniet

Functio-

Functionem fractam genuinam adhibere: si enim fractio spuria accipiatur, tum pars integra in ea contenta ad Seriem accedet, atque in illis terminis, quos vel auget vel minuit, legem progressionis interrumpet. Exempli gratia hæc fractio spuria $\frac{1+2z-z^3}{1-z-zz}$, præbebit hanc Seriem $1+3z+4zz+6z^3+10z^4+16z^5+26z^6+42z^7+$ &c. ubi a lege, qua quivis coëfficiens est summa duorum præcedentium, terminus quartus $6z^3$ excipitur.

64. Peculiarem contemplationem Series recurrentes merentur, si denominator fractionis, unde oriuntur, fuerit potestas. Sic, si ista fractio $\frac{a+bz}{(1-\alpha z)^2}$ in Seriem resolvatur, prodit

$$\begin{array}{llllll} a & +2\alpha a\,z & +3\alpha^2 a\,z^2 & +4\alpha^3 a\,z^3 & +5\alpha^4 a\,z^4 & +\ \&c. \\ & +\ b & +2\alpha b & +3\alpha^2 b & +4\alpha^3 b & \end{array}$$

in qua coëfficiens potestatis z^n erit $(n+1)\alpha^n a+n\alpha^{n-1}b$. Erit tamen hæc Series recurrens, quia quilibet terminus ex duobus præcedentibus determinatur, cujus determinationis lex perspicitur ex denominatore evoluto $1-2\alpha z+\alpha\alpha zz$. Si ponatur $\alpha=1$ & $z=1$, abit Series in progressionem arithmeticam generalem $a+(2a+b)+(3a+2b)+(4a+3b)+$ &c. cujus differentiæ sunt constantes. Omnis ergo progressio arithmetica est Series recurrens: si enim sit
$A+B+C+D+E+F+$ &c. progressio arithmetica, erit $C=2B-A$; $D=2C-B$; $E=2D-C$, &c.

65. Deinde hæc fractio $\frac{a+bz+czz}{(1-\alpha z)^3}$ ob $\frac{1}{(1-\alpha z)^3}=(1-\alpha z)^{-3}$ $=1+3\alpha z+6\alpha^2z^2+10\alpha^3z^3+15\alpha^4z^4+$ &c. transmutabitur in hanc Seriem infinitam:

$$\begin{array}{llllll} a & +3\alpha a\,z & +6\alpha^2 a\,z^2 & +10\alpha^3 a\,z^3 & +15\alpha^4 a\,z^4 & +\ \&c. \\ & +\ b & +3\alpha b & +6\alpha^2 b & +10\alpha^3 b & \\ & & +\ c & +3\alpha c & +6\alpha^2 c & \end{array}$$

 in qua poteſtas z^n coëfficientem habebit $\frac{(n+1)(n+2)}{1.\ 2}\alpha^n a + \frac{n(n+1)}{1.\ 2}\alpha^{n-1}b + \frac{(n-1)n}{1.\ 2}\alpha^{n-2}c$. Quod ſi autem ponatur $\alpha = 1$ & $z = 1$, Series hæc abibit in progreſſionem generalem ſecundi ordinis, cujus differentiæ ſecundæ ſunt conſtantes. Deſignet $A + B + C + D + E +$ &c. hujuſmodi progreſſionem, erit ea ſimul Series recurrens, cujus quilibet terminus ex tribus antecedentibus ita determinatur ut ſit $D = 3C - 3B + A$; $E = 3D - 3C + B$; $F = 3E - 3D + C$ &c. Cum igitur terminorum in progreſſione arithmetica procedentium ſecundæ differentiæ quoque ſint æquales, nempe $= 0$, hæc proprietas quoque ad progreſſiones arithmeticas extenditur.

66. Simili modo hæc fractio $\frac{a + bz + czz + dz^3}{(1 - \alpha z)^4}$ dabit Seriem infinitam, in qua poteſtas ipſius z quæcunque z^n hunc habebit coëfficientem $\frac{(n+1)(n+2)(n+3)}{1.\ 2.\ 3}\alpha^n a + \frac{n(n+1)(n+2)}{1.\ 2.\ 3}\alpha^{n-1}b + \frac{(n-1)n(n+1)}{1.\ 2.\ 3}\alpha^{n-2}c + \frac{(n-2)(n-1)n}{1.\ 2.\ 3}\alpha^{n-3}d$: poſito ergo $\alpha = 1$ & $z = 1$; hæc Series in ſe complectetur omnes progreſſiones algebraicas tertii ordinis, quarum differentiæ tertiæ ſunt conſtantes: omnes ergo hujus ordinis progreſſiones, cujuſmodi ſit $A + B + C + D + E + F +$ &c. erunt ſimul recurrentes ex denominatore $1 - 4z + 6zz - 4z^3 + z^4$ ortæ; unde erit $E = 4D - 6C + 4B - A$; $F = 4E - 6D + 4C - B$ &c., quæ proprietas ſimul in omnes progreſſiones inferiorum ordinum competit.

67. Hoc modo oſtendentur omnes progreſſiones algebraicas cujuscunque ordinis, quæ tandem ad differentias conſtantes deducunt, eſſe Series recurrentes, quarum lex definiatur ex denominatore $(1 - z)^n$, exiſtente n numero majore quam is, qui ordinem progreſſionis indicat. Cum igitur $a^m + (a+b)^m +$

$(a+2b)$

$(a+2b)^m + (a+3b)^m +$ &c. exhibeat progressio- CAP. IV.
nem ordinis m; erit ob naturam Serierum recurrentium

$$0 = a^m - \frac{n}{1}(a+b)^m + \frac{n(n-1)}{1.\ 2}(a+2b)^m$$

$$- \frac{n(n-1)(n-2)}{1.\ 2.\ 3}(a+3b)^m + \ldots\ldots\ldots\ldots \pm$$

$\frac{n}{1}(a+(n-1)b)^m \mp (a+nb)^m$; ubi signa superiora valent si n sit numerus par, inferiora autem si n sit numerus impar. Hæc ergo æquatio semper est vera si fuerit n numerus integer major quam m. Hinc ergo intelligitur quam late pateat doctrina de Seriebus recurrentibus.

68. Si denominator fuerit potestas non binomii sed multinomii, natura Seriei quoque alio modo explicari potest. Sit nempe hæc fractio $\frac{1}{(1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\&c.)^{m+1}}$ proposita, erit Series infinita hinc nata

$$1+\frac{(m+1)}{1}\alpha z+\frac{(m+1)(m+2)}{1.\ 2}\alpha^2 z^2+\frac{(m+1)(m+2)(m+3)}{1.\ 2.\ 3}\alpha^3 z^3$$
$$+\frac{(m+1)}{1}\beta z^2+\frac{(m+1)(m+2)}{1.\ 2}2\alpha\beta z^3+\&c.$$
$$+\frac{(m+1)}{1}\gamma z^3$$

Ad naturam hujus Seriei penitius inspiciendam, exponatur hæc Series per litteras generales hoc modo:

$1+Az+Bz^2+Cz^3+\ldots+Kz^{n-3}+Lz^{n-2}+Mz^{n-1}+Nz^n+$ &c., ac quilibet coëfficiens N ex tot procedentibus, quot sunt litteræ $\alpha.\ \beta.\ \gamma.\ \delta.$ &c. ita determinabitur ut sit: $N = \frac{m+n}{n}\alpha M+\frac{2m+n}{n}\beta L+\frac{3m+n}{n}\gamma K+\frac{4m+n}{n}\delta I+$ &c. quæ lex continuationis etsi non est constans, sed ab exponente potestatis z pendet, tamen eidem Seriei alia convenit lex progressionis constans, quam denominator evolutus præbet, na-

 turæ

LIB. I. turæ Serierum recurrentium consentaneam. Illa vero lex non constans tantum locum habet si numerator fractionis fuerit unitas seu quantitas constans; si enim quoque aliquot potestates ipsius z contineret, tum illa lex multo magis fieret complicata, id quod post tradita calculi differentialis principia facilius patebit.

69. Quoniam hactenus posuimus primum denominatoris terminum constantem non esse $= 0$, ejusque loco unitatem collocavimus; nunc videamus cujusmodi Series oriantur, si in denominatore terminus constans evanescat. His casibus ergo Functio fracta hujusmodi formam habebit $\frac{a + bz + czz + \&c.}{z(1 - \alpha z - \beta zz - \&c.)}$, convertatur ergo, neglecto denominatoris Factore z, reliqua fractio $\frac{a + bz + czz + \&c.}{1 - \alpha z - \beta z^2 - \&c.}$ in Seriem recurrentem $A + Bz + Cz^2 + Dz^3 + \&c.$ atque manifestum est fore $\frac{a + bz + czz + \&c.}{z(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)} = \frac{A}{z} + B + Cz + Dz^2 + Ez^3 + \&c.$ Simili modo erit $\frac{a + bz + cz^2 + \&c.}{z^2(1 - \alpha z - \beta z^2 - \&c.)} = \frac{A}{zz} + \frac{B}{z} + C + Dz + Ez^2 + \&c.$, atque generatim erit $\frac{a + bz + cz^2 + \&c.}{z^m(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)} = \frac{A}{z^m} + \frac{B}{z^{m-1}} + \frac{C}{z^{m-2}} + \frac{D}{z^{m-3}} + \&c.$ quicunque numerus fuerit exponens m.

70. Quoniam per substitutionem loco z alia variabilis x in Functionem fractam introduci, hocque pacto Functio fracta quævis in innumerabiles formas diversas transmutari potest; hoc modo eadem Functio fracta infinitis modis per Series recurrentes explicari poterit. Sit scilicet proposita hæc fractio $y = \frac{1 + z}{1 - z - zz}$ & per Seriem recurrentem $y = 1 + 2z + 3z^2 + 5z^3 + 8z^4 + \&c.$: ponatur $z = \frac{1}{x}$ erit y $=$

$= \frac{xx + x}{xx - x - 1} = \frac{-x(1+x)}{1 + x - xx}$. Jam $\frac{1 + x}{1 + x - xx} = 1 +$ CAP. IV.
$0x + xx - x^3 + 2z^4 - 3x^5 + 5x^6 -$ &c.; unde erit $y = -x + 0x^2 - x^3 + x^4 - 2x^5 + 3x^6 - 5x^7 +$ &c. Vel ponatur $z = \frac{1 - x}{1 + x}$, erit $y = \frac{-2 - 2x}{1 - 4x - xx}$; unde fit $y = -2 - 10x - 42xx - 178x^3 - 754x^4 -$ &c. cujusmodi Series recurrentes pro y innumerabiles inveniri possunt.

71. Functiones irrationales ex hoc theoremate universali in Series infinitas transformari solent, quod sit $(P + Q)^{\frac{m}{n}}$ $= P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m-n}{n}} Q + \frac{m(m-n)}{n.\ 2n} P^{\frac{m-2n}{n}} Q^2 +$ $\frac{m(m-n)(m-2n)}{n.\ 2n.\ 3n} P^{\frac{m-3n}{n}} Q^3 +$ &c.: hi enim termini, nisi fuerit $\frac{m}{n}$ numerus integer affirmativus, in infinitum excurrunt. Sic erit pro m & n numeros definitos scribendo.

$$(P + Q)^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{1}{2} P^{-\frac{1}{2}} Q - \frac{1.1}{2.4} P^{-\frac{3}{2}} Q^2 + \frac{1.1.3}{2.4.6} P^{-\frac{5}{2}} Q^3 - \text{\&c.}$$

$$(P + Q)^{-\frac{1}{2}} = P^{-\frac{1}{2}} - \frac{1}{2} P^{-\frac{3}{2}} Q + \frac{1.3}{2.4} P^{-\frac{5}{2}} Q^2 - \frac{1.3.5}{2.4.6} P^{-\frac{7}{2}} Q^3 + \text{\&c.}$$

$$(P + Q)^{\frac{1}{3}} = P^{\frac{1}{3}} + \frac{1}{3} P^{-\frac{2}{3}} Q - \frac{1.2}{3.6} P^{-\frac{5}{3}} Q^2 + \frac{1.2.5}{3.6.19} P^{-\frac{8}{3}} Q^3 - \text{\&c.}$$

(P

LIB. I.

$$(P+Q)^{-\frac{1}{3}} = P^{-\frac{1}{3}} - \frac{1}{3} P^{-\frac{4}{3}} Q + \frac{1.4}{3.6} P^{-\frac{7}{3}} Q^2 - \frac{1.4.7}{3.6.9} P^{-\frac{10}{3}} Q^3 + \&c.$$

$$(P+Q)^{\frac{2}{3}} = P^{\frac{2}{3}} + \frac{2}{3} P^{-\frac{1}{3}} Q - \frac{2.1}{3.6} P^{-\frac{4}{3}} Q^2 + \frac{2.1.4}{3.6.9} P^{-\frac{7}{3}} Q^3 - \&c.$$

&c.

72. Hujuſmodi ergo Serierum termini ita progrediuntur ut quilibet ex antecedente formari poſſit: ſit enim Seriei, quæ ex $(P+Q)^{\frac{m}{n}}$ naſcitur, terminus quilibet $= MP^{\frac{m-kn}{n}} Q^k$ erit ſequens $= \frac{m-kn}{(k+1)n} MP^{\frac{m-(k+1)n}{n}} Q^{k+1}$. Notandum autem eſt in quovis termino ſequente exponentem ipſius P unitate decreſcere, contra vero exponentem ipſius Q unitate creſcere. Quo autem hæc facilius ad quemvis caſum accommodentur, forma generalis $(P+Q)^{\frac{m}{n}}$ ita exponi poteſt $P^{\frac{m}{n}} (1 + \frac{Q}{P})^{\frac{m}{n}}$: evoluta enim formula $(1 + \frac{Q}{P})^{\frac{m}{n}}$ Serieque reſultante per $P^{\frac{m}{n}}$ multiplicata, prodibit ipſa Series ante data. Tum vero ſi m non ſolum numeros integros denotet, ſed etiam fractos, pro n tuto unitas collocari poterit. Quibus factis, ſi pro $\frac{Q}{P}$, quæ eſt Functio ipſius z, ponatur Z, habebitur

$$(1+Z)^m = 1 + \frac{m}{1} Z + \frac{m(m-1)}{1.2} Z^2 + \frac{m(m-1)(m-2)}{1.2.3} Z^3$$

$+ \&c.$ Ad ſequentes progreſſionum leges autem obſervandas conveniet hanc formulæ generalis in Seriem converſionem notaſſe

taſſe $(1+Z)^{m-1} = 1 + \frac{(m-1)}{1} Z + \frac{(m-1)(m-2)}{1.\ 2} Z^2 + \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3} Z^3 + \&c.$ CAP. IV.

73. Sit igitur primum $Z = \alpha z$, eritque $(1+\alpha z)^{m-1} = 1 + \frac{m-1}{1} \alpha z + \frac{(m-1)(m-2)}{1\ \ 2} \alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3} \alpha^3 z^3 + \&c.$ Scribatur pro hac Serie iſta forma generalis

$$1 + Az + Bz^2 + Cz^3 + \ldots\ldots + Mz^{n-1} + Nz^n + \&c.$$

atque quilibet coëfficiens N ex præcedente M ita determinabitur ut ſit $N = \frac{m-n}{n} \alpha M$. Sic, poſito $n=1$, cum ſit $M=1$, erit $N = A = \frac{m-1}{1} \alpha$; tum facto $n=2$, ob $M = A = \frac{m-1}{1} \alpha$, erit $N = B = \frac{m-2}{2} \alpha M = \frac{(m-1)(m-2)}{1.\ 2} \alpha^2$; ſimilique modo porro $C = \frac{m-3}{3} \alpha B = \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3} \alpha^3$, uti Series ante inventa declarat..

74. Sit $Z = \alpha z + \beta zz$, eritque $(1 + \alpha z + \beta zz)^{m-1} = 1 + \frac{(m-1)}{1}(\alpha z + \beta zz) + \frac{(m-1)(m-2)}{1.\ 2}(\alpha z + \beta zz)^2 + \&c.$ Quod ſi ergo termini ſecundum poteſtates ipſius z diſponantur erit $(1 + \alpha z + \beta zz)^{m-1} =$

$$1 + \frac{(m-1)}{1} \alpha z + \frac{(m-1)(m-2)}{1.\ 2} \alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3} \alpha^3 z^3 + \&c.$$
$$+ \frac{(m-1)}{1} \beta z^2 + \frac{(m-1)(m-2)}{1.\ 2} 2\alpha\beta z^3 + \&c.$$

LIB. I. Scribatur pro hac Serie ista forma generalis:

$$1+Az+Bz^2+Cz^3+\ldots+Lz^{n-2}+Mz^{n-1}+Nz^n+\&c.$$

atque quilibet coëfficiens ex duobus antecedentibus ita definietur ut sit $N=\frac{m-n}{n}\alpha M+\frac{2m-n}{n}\beta L$, unde omnes termini ex primo, qui est 1, definiri poterunt. Erit nempe

$$A=\frac{m-1}{1}\alpha;$$
$$B=\frac{(m-2)}{2}\alpha A+\frac{(2m-2)}{2}\beta$$
$$C=\frac{(m-3)}{3}\alpha B+\frac{(2m-3)}{3}\beta A$$
$$D=\frac{(m-4)}{4}\alpha C+\frac{(2m-4)}{4}\beta B$$
&c.

75. Si fuerit $Z=\alpha z+\beta z^2+\gamma z^3$, erit $(1+\alpha z+\beta z^2+\gamma z^3)^{m-1}=1+\frac{(m-1)}{1}(\alpha z+\beta z^2+\gamma z^3)+\frac{(m-1)(m-2)}{1.\ 2}(\alpha z+\beta z^2+\gamma z^3)^2+\&c.$, quæ expressio, si omnes termini secundum potestates ipsius z ordinentur, abibit in hanc Seriem:

$$1+\frac{(m-1)}{1}\alpha z+\frac{(m-1)(m-2)}{1.\ 2}\alpha^2 z^2+\frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3}\alpha^3 z^3$$
$$+\frac{(m-1)}{1}\beta z^2+\frac{(m-1)(m-2)}{1.\ 2}2\alpha\beta z^3+\&c.$$
$$+\frac{(m-1)}{1}\gamma z^3$$

cujus lex progressionis ut melius patescat, ponatur ejus loco $1+Az+Bz^2+Cz^3+\ldots+Kz^{n-3}+Lz^{n-2}+Mz^{n-1}+Nz^n$, cujus Seriei quilibet coëfficiens ex tribus antecedentibus ita determinatur ut sit $N=\frac{(m-n)}{n}\alpha M+\frac{(2m-n)}{n}\beta L+\frac{(3m-n)}{n}\gamma K$.

Cum

Cum igitur primus terminus ſit $= 1$, & antecedentes nulli, erit

$$A = \frac{m-1}{1}\alpha$$
$$B = \frac{(m-2)}{2}\alpha A + \frac{(2m-2)}{2}\beta$$
$$C = \frac{(m-3)}{3}\alpha B + \frac{(2m-3)}{3}\beta A + \frac{(3m-3)}{3}\gamma$$
$$D = \frac{(m-4)}{4}\alpha C + \frac{(2m-4)}{4}\beta B + \frac{(3m-4)}{4}\gamma A$$
$$E = \frac{(m-5)}{5}\alpha D + \frac{(2m-5)}{5}\beta C + \frac{(3m-5)}{5}\gamma B$$
&c.

76. Generaliter ergo ſi ponatur $(1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \&c.)^{m-1} = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \&c.$, hujus Seriei ſinguli termini ita ex præcedentibus definientur, ut ſit

$$A = \frac{m-1}{1}\alpha$$
$$B = \frac{(m-2)}{2}\alpha A + \frac{(2m-2)}{2}\beta$$
$$C = \frac{(m-3)}{3}\alpha B + \frac{(2m-3)}{3}\beta A + \frac{(3m-3)}{3}\gamma$$
$$D = \frac{(m-4)}{4}\alpha C + \frac{(2m-4)}{4}\beta B + \frac{(3m-4)}{4}\gamma A + \frac{(4m-4)}{4}\delta$$
$$E = \frac{(m-5)}{5}\alpha D + \frac{(2m-5)}{5}\beta C + \frac{(3m-5)}{5}\gamma B + \frac{(4m-5)}{5}\delta A + \frac{(5m-5)}{5}\varepsilon$$
&c.

quilibet ſcilicet terminus per tot præcedentes determinatur, quot habentur litteræ α, β, γ, δ, &c. in Functione ipſius z cujus poteſtas in Seriem convertitur. Ceterum ratio hujus legis convenit cum ea, quam ſupra §. 68. ubi ſimilem formam $(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)^{-m-1}$ in Seriem infini-

 tam

LIB. I. tam resolvimus; si enim loco m scribatur $-m$ atque litteræ α, β, γ, δ, &c. negative accipiantur, Series inventæ prorsus congruent. Interim hoc loco non licet rationem hujus progressionis legis a priori demonstrare, id quod per principia calculi differentialis demum commode fieri poterit; interea ergo sufficiet veritatem per applicationem ad omnis generis exempla comprobasse.

CAPUT V.

De Functionibus duarum pluriumve variabilium.

77. Quanquam plures hactenus quantitates variabiles sumus contemplati, tamen eæ ita erant comparatæ, ut omnes unius essent Functiones, unaque determinata reliquæ simul determinarentur. Nunc autem ejusmodi considerabimus quantitates variabiles, quæ a se invicem non pendeant, ita ut quamvis uni determinatus valor tribuatur, reliquæ tamen nihilominus maneant indeterminatæ ac variabiles. Ejusmodi ergo quantitates variabiles, cujusmodi sint x, y, z, ratione significationis convenient, cum quælibet omnes valores determinatos in se complectatur; at, si inter se comparentur maxime erunt diversæ, cum, licet pro una z valor quicunque determinatus substituatur, reliquæ tamen x & y æque late pateant, atque ante. Discrimen ergo inter quantitates variabiles a se pendentes, & non pendentes in hoc versatur, ut priori casu, si una determinetur, simul reliquæ determinentur; posteriori vero determinatio unius significationes reliquarum minime restringat.

78. *Functio ergo duarum pluriumve quantitatum variabilium,* x, y, z, *est expressio quomodocunque ex his quantitatibus composita.*

Ita erit $x^3 + xyz + az^2$ Functio quantitatum variabilium trium x, y, z. Hæc ergo Functio, si una determinetur variabilis,

riabilis, puta z, hoc eſt ejus loco conſtans numerus ſubſtitua- CAP. V.
tur, manebit adhuc quantitas variabilis, ſcilicet Functio ipſarum x & y. Atque ſi, præter z, quoque y determinetur, tum erit adhuc Functio ipſius x. Hujuſmodi ergo plurium variabilium Functio non ante valorem determinatum obtinebit, quam ſingulæ quantitates variabiles fuerint determinatæ. Cum igitur una quantitas variabilis infinitis modis determinari poſſit, Functio duarum variabilium, quia pro quavis determinatione unius infinitas determinationes ſuſcipere poteſt, omnino infinities infinitas determinationes admittet. Atque in Functione trium variabilium numerus determinationum erit adhuc infinities major; ſicque porro creſcet pro pluribus variabilibus.

79. *Hujuſmodi Functiones plurium variabilium perinde atque Functiones unius variabilis, commodiſſime dividuntur in algebraicas ac tranſcendentes.*

Quarum illæ ſunt, in quibus ratio compoſitionis in ſolis Algebræ operationibus eſt poſita; hæ vero, in quarum formationem quoque operationes tranſcendentes ingrediuntur. In his denuo ſpecies notari poſſent, prout operationes tranſcendentes vel omnes quantitates variabiles implicant, vel aliquot, vel tantum unicam. Sic iſta expreſſio $zz + y \log. z$, quia Logarithmus ipſius z ineſt, erit quidem Functio tranſcendens ipſarum y & z, verum ideo minus tranſcendens eſt putanda, quod ſi variabilis z determinetur, ſuperſit Functio algebraica ipſius y. Interim tamen non expedit hujuſmodi ſubdiviſionibus tractationem amplificari.

80. *Functiones deinde algebraicæ ſubdividuntur in rationales & irrationales; rationales autem porro in integras ac fractas.*

Ratio harum denominationum ex Capite primo jam abunde intelligitur. Functio ſcilicet rationalis omnino eſt libera ab omni irrationalitate quantitates variabiles, quarum Functio dicitur, afficiente; hæcque erit integra ſi nullis fractionibus inquinetur, contra vero fracta. Sic Functionis integræ duarum variabilium y & z hæc erit forma generalis: $\alpha + \beta y + \gamma z + \delta y^2 + \varepsilon yz + \zeta z^2 + \eta z^3 + \theta y^2 z + \iota y z^2 + \varkappa z^3 +$ &c. Quod

ſi ergo P & Q denotent hujuſmodi Functiones integras, ſive duarum ſive plurium variabilium, erit $\frac{P}{Q}$ forma generalis Functionum fractarum. Functio denique irrationalis eſt vel explicita, vel implicita; illa per ſigna radicalia jam penitus eſt evoluta, hæc autem per æquationem irreſolubilem exhibetur: ſic V erit Functio implicita irrationalis ipſarum y & z, ſi fuerit $V^5 = (ayz + z^3) V^2 + (y^4 + z^4) V + y^5 + 2ayz^3 + z^5$.

81. *Multiformitas deinde in his Functionibus æque notari debet, atque in iis, quæ ex unica variabili conſtant.*

Sic Functiones rationales erunt uniformes, quia ſingulis quantitatibus variabilibus determinatis, unicum valorem determinatum exibent. Denotent P, Q, R, S, &c. Functiones rationales ſeu uniformes variabilium x, y, z, eritque V Functio biformis earundem variabilium, ſi fuerit $V^2 - PV + Q = 0$; quicunque enim valores determinati quantitatibus x, y, & z tribuuntur, Functio V non unum ſed duplicem perpetuo habebit valorem determinatum. Simili modo erit V Functio triformis ſi fuerit $V^3 - PV^2 + QV - R = 0$: atque Functio quadriformis ſi fuerit $V^4 - PV^3 + QV^2 - RV + S = 0$: hocque modo ratio Functionum multiformium ulteriorum erit comparata.

82. Quemadmodum ſi Functio unius variabilis z nihilo æqualis ponitur, quantitas variabilis z valorem conſequitur determinatum vel ſimplicem vel multiplicem; ita ſi Functio duarum variabilium y & z nihilo æqualis ponitur, tum altera variabilis per alteram definitur, ejuſque ideo Functio evadit, cum ante a ſe mutuo non penderent. Simili modo ſi Functio trium variabilium x, y, z, nihilo æqualis ſtatuatur, tum una variabilis per duas reliquas definitur, earumque Functio exiſtit. Idem evenit ſi Functio non nihilo ſed quantitati conſtanti vel etiam alii Functioni æqualis ponatur; ex omni enim æquatione, quotcunque variabiles involvat, ſemper una variabilis per reliquas definitur earumque fit Functio; duæ autem æquationes

tiones diversæ inter easdem variabiles ortæ binas per reliquas definient, atque ita porro. Cap. V.

83. *Functionum autem duarum pluriumve variabilium divisio maxime notatu digna est in homogeneas & heterogeneas.*

Functio homogenea est per quam ubique idem regnat variabilium numerus dimensionum: Functio autem heterogenea est, in qua diversi occurrunt dimensionum numeri. Censetur vero unaquæque variabilis unam dimensionem constituere; quadratum uniuscujusque atque productum ex duabus, duas; productum ex tribus variabilibus, sive iisdem sive diversis, tres & ita porro; quantitates autem constantes ad dimensionum numerationem non admittuntur. Ita in his formulis αy; βz, unica dimensio inesse dicitur; in his vero αy^2; βyz; γz^2 duæ insunt dimensiones: in his αy^3; $\beta y^2 z$; γyz^2; δz^3, tres; in his vero αy^4; $\beta y^3 z$; $\gamma y^2 z^2$; δyz^3; εz^4; quatuor, sicque porro.

84. Applicemus primum hanc distinctionem ad Functiones integras, atque duas tantum variabiles inesse ponamus, quoniam plurium par est ratio.

Functio igitur integra erit homogenea in cujus singulis terminis idem existit dimensionum numerus.

Subdividentur ergo hujusmodi Functiones commodissime secundum numerum dimensionum, quem variabiles in ipsis ubique constituunt. Sic erit $\alpha y + \beta z$ forma generalis Functionum integranum unius dimensionis: hæc vero expressio $\alpha y^2 + \beta yz + \gamma z^2$ erit forma generalis Functionum duarum dimensionum, tum forma generalis Functionum trium dimensionum erit: $\alpha y^3 + \beta y^2 z + \gamma yz^2 + \delta z^3$; quatuor dimensionum vero hæc: $\alpha y^4 + \beta y^3 z + \gamma y^2 z^2 + \delta yz^3 + \varepsilon z^4$; & ita porro. Ad analogiam igitur erit quantitas constans sola α Functio nullius dimensionis.

85. *Functio porro fracta erit homogenea, si ejus Numerator ac Denominator fuerint Functiones homogeneæ.*

Sic hæc Fractio $\frac{ayy + bzz}{\alpha y + \beta z}$ erit Functio homogenea ipsa-

rum

LIB. I. rum y & z; numerus dimensionum autem habebitur, si a numero dimensionum Numeratoris subtrahatur numerus dimensionum Denominatoris: atque ob hanc rationem Fractio allata erit Functio unius dimensionis. Hæc vero Fractio $\frac{y^5 + z^5}{yy + zz}$ erit Functio trium dimensionum. Quando ergo in Numeratore ac Denominatore idem dimensionum numerus inest, tum Fractio erit Functio nullius dimensionis, uti evenit in hac Fractione $\frac{y^3 + z^3}{yyz}$, vel etiam in his $\frac{y}{z}$; $\frac{\alpha zz}{yy}$; $\frac{\beta y^3}{z^3}$. Quod si igitur in Denominatore plures sint dimensiones quam in Numeratore, numerus dimensionum Fractionis erit negativus: sic $\frac{y}{zz}$ erit Functio -1 dimensionis: $\frac{y+z}{y^4+z^4}$ erit Functio -3 dimensionum: $\frac{1}{y^5 + ayz^4}$ erit Functio -5 dimensionum, quia in Numeratore nulla inest dimensio. Ceterum sponte intelligitur plures Functiones homogeneas, in quibus singulis idem regnat dimensionum numerus, sive additas sive subtractas præbere Functionem quoque homogeneam ejusdem dimensionum numeri. Sic hæc expressio $\alpha y + \frac{\beta zz}{y} + \frac{\gamma y^4 - \delta z^4}{yyz + yzz}$ erit Functio unius dimensionis: hæc autem $\alpha + \frac{\beta y}{z} + \frac{\gamma zz}{yy} + \frac{yy + zz}{yy - zz}$ erit Functio nullius dimensionis.

86. Natura Functionum homogenearum quoque ad expressiones irrationales extenditur. Si enim fuerit P Functio quæcunque homogenea, puta n dimensionum, tum $\sqrt{P}$ erit Functio $\frac{1}{2}n$ dimensionum; $\sqrt[3]{P}$ erit Functio $\frac{1}{3}n$ dimensionum, & generatim $P^{\frac{\mu}{\nu}}$ erit Functio $\frac{\mu}{\nu}n$ dimensionum. Sic $\sqrt{(yy+zz)}$ erit Functio unius dimensionis; $\sqrt[3]{(y^9 + z^9)}$ erit Functio trium dimensionum: $(yz + zz)^{\frac{3}{4}}$ erit Functio $\frac{3}{2}$ dimensionum: atque

que $\frac{yy+zz}{\sqrt{(y^4+z^4)}}$ erit Functio nullius dimensionis. His ergo cum præcedentibus conjunctis intelligetur hæc expressio $\frac{1}{y}$ $+\frac{y\sqrt{(yy+zz)}}{z^3}-\frac{y}{\sqrt[3]{(y^6-z^6)}}+\frac{y\sqrt{z}}{zz\sqrt{y}+\sqrt{(y^5+z^5)}}$ esse Functio homogenea -1 dimensionis.

87. Utrum Functio irrationalis implicita sit homogenea necne, ex his facile colligi potest. Sit V hujusmodi Functio implicita ac $V^3+PV^2+QV+R=0$, existentibus P, Q & R Functionibus ipsarum y & z. Primum igitur patet V Functionem homogeneam esse non posse, nisi P, Q, & R sint Functiones homogeneæ. Præterea vero si ponamus V esse Functionem n dimensionum, erit V^2 Functio $2n$, & V^3 Functio $3n$ dimensionum; cum igitur ubique idem debeat esse numerus dimensionum, oportet, ut P sit Functio n dimensionum, Q Functio $2n$ dimensionum, & R Functio $3n$ dimensionum. Si ergo vicissim litteræ P, Q, R Functiones homogeneæ respective n, $2n$, $3n$ dimensionum, hinc concludetur fore V Functionem n dimensionum. Ita si fuerit $V^5+(y^4+z^4)V^3+\alpha y^8V-z^{10}=0$ erit V Functio homogenea duarum dimensionum, ipsarum y & z.

88. *Si fuerit* V *Functio homogenea* n *dimensionum ipsarum* y *&* z, *in eaque ponatur ubique* $y=uz$, *Functio V abibit in productum ex potestate* z^n *in Functionem quandam variabilis* u.

Per hanc enim substitutionem $y=uz$, in singulos terminos tantæ inducentur potestates ipsius z, quantæ ante inerant ipsius y. Cum igitur in singulis terminis dimensiones ipsarum y & z conjunctim æquassent numerum n, nunc sola variabilis z ubique habebit n dimensiones, ideoque ubique inerit ejus potestas z^n. Per hanc ergo potestatem Functio V fiet divisibilis & quotus erit Functio variabilem tantum u involvens. Hoc primum patebit in Functionibus integris; si enim sit $V=\alpha y^3+\beta y^2z+\gamma yz^2+\delta z^3$, posito $y=uz$, fiet $V=z^3$

 $(\alpha u^3+$

$(\alpha u^3 + \beta u^2 + \gamma u + \delta)$. Deinde vero idem manifeſtum eſt in fractis: ſit enim $V = \frac{\alpha y + \beta z}{yy + zz}$, nempe Functio -1 dimenſionis, facto $y = uz$ fiet $V = z^{-1}\left(\frac{\alpha u + \beta}{uu + 1}\right)$. Neque etiam Functiones irrationales hinc excipiuntur, ſi enim ſit $V = \frac{y + \sqrt{(yy + zz)}}{z\sqrt{(y^3 + z^3)}}$, quæ eſt Functio $-\frac{3}{2}$ dimenſionum; poſito $y = uz$, prodibit $V = z^{-\frac{3}{2}}\left(\frac{u + \sqrt{(uu + 1)}}{\sqrt{(u^3 + 1)}}\right)$. Hoc itaque modo Functiones homogeneæ duarum tantum variabilium reducentur ad Functiones unius variabilis; neque enim poteſtas ipſius z, quia eſt Factor, Functionem illam ipſius u inquinat.

89. *Functio ergo homogenea* V *duarum variabilium* y *&* z *nullius dimenſionis, poſito* y = u z, *tranſmutabitur in Functionem unicæ variabilis* u *puram.*

Cum enim numerus dimenſionum ſit nullus, Poteſtas ipſius z, quæ Functionem ipſius u multiplicabit, erit $z^0 = 1$; hocque caſu variabilis z prorſus ex computo egredietur. Ita ſi fuerit $V = \frac{y + z}{y - z}$, facto $y = uz$, orietur $V = \frac{u + 1}{u - 1}$: atque in irrationalibus ſi ſit $V = \frac{y - \sqrt{(yy - zz)}}{z}$ poſito $y = uz$ erit $V = u - \sqrt{(uu - 1)}$.

90. *Functio integra homogenea duarum variabilium* y *&* z, *reſolvi poterit in tot Factores ſimplices forma* $\alpha y + \beta z$, *quot habuerit dimenſiones.*

Cum enim Functio ſit homogenea, poſito $y = uz$, tranſibit in productum ex z^n in Functionem quandam ipſius u integram, quæ Functio propterea in Factores ſimplices formæ $\alpha u + \beta$ reſolvi poterit. Multiplicentur ſinguli Factores hi per z, eritque uniuſcujuſque forma $\alpha uz + \beta z = \alpha y + \beta z$ ob $uz = y$. Propter multiplicatorem autem z^n, tot hujuſmodi Factores naſcentur quot exponens n contineat unitates; Factores autem

hi

hi ſimplices erunt vel reales vel imaginarii, hoc eſt coëfficientes α, & β erunt vel reales vel imaginarii. Cap. V.

Ex hoc itaque ſequitur Functionem duarum dimenſionum $ayy + byz + czz$ duos habere Factores ſimplices formæ $\alpha y + \beta z$; Functio autem $ay^3 + by^2z + cyz^2 + dz^3$ habebit tres Factores ſimplices formæ $\alpha y + \beta z$; ſicque porro Functionum homogenearum integrarum, quæ plures habent dimenſiones, natura erit comparata.

91. Quemadmodum ergo hæc expreſſio $\alpha y + \beta z$ continet formam generalem Functionum integrarum unius dimenſionis - ita $(\alpha y + \beta z)(\gamma y + \delta z)$ erit forma generalis Functionum in, tegrarum duarum dimenſionum: atque in hac forma $(\alpha y + \beta z)$ $(\gamma y + \delta z)(\varepsilon y + \xi z)$ continebuntur omnes Functiones integræ trium dimenſionum, ſicque omnes Functiones integræ homogeneæ per producta ex tot hujuſmodi Factoribus $\alpha y + \beta z$ exhiberi poterunt, quot Functiones illæ contineant dimenſiones. Iſti autem Factores eodem modo per reſolutionem æquationum reperiuntur, quo ſupra Factores ſimplices Functionum integrarum unius variabilis invenire docuimus. Ceterum hæc proprietas Functionum homogenearum duarum variabilium non extenditur ad Functiones homogeneas trium pluriumve variabilium: forma enim generalis hujuſmodi Functionum duarum tantum dimenſionum, quæ eſt $ayy + byz + cyx + dxy + exx + fzz$ generaliter non reduci poteſt ad hujuſmodi productum $(\alpha y + \beta z + \gamma x)(\delta y + \varepsilon z + \xi x)$; multoque minus Functiones plurium dimenſionum ad hujuſmodi producta revocari poſſunt.

92. Ex his, quæ de Functionibus homogeneis ſunt dicta, ſimul intelligitur, quid ſit Functio heterogenea: in cujus ſcilicet terminis non ubique idem dimenſionum numerus deprehenditur. Poſſunt autem Functiones heterogeneæ ſubdividi pro multiplicitate dimenſionum, quæ in ipſis occurrunt. Sic Functio bifida erit, in qua duplex dimenſionum numerus occurrit, eritque adeo aggregatum duarum Functionum homo-

LIB. I. genearum, quarum numeri dimensionum differunt; ita $y^5 + 2y^3z^2 + yy + zz$ erit Functio bifida, quia partim quinque, partim duas continet dimensiones. Functio autem trifida est, in qua tres diversi dimensionum numeri insunt, seu quæ in tres Functiones homogeneas distribui possunt, uti $y^6 + y^2z^2 + z^4 + y - z$.

Præterea autem dantur Functiones heterogeneæ fractæ vel irrationales tantopere permixtæ, quæ in Functiones homogeneas resolvi non possunt, cujusmodi sunt $\frac{y^3 + ayz}{by + zz}$, $\frac{a + \sqrt{(yy + zz)}}{yy - bz}$.

93. Interdum Functio heterogenea ope substitutionis idoneæ, vel loco unius vel utriusque variabilis factæ, ad homogeneam reduci potest; quod quibus casibus fieri queat, non tam facile indicare licet. Sufficiet ergo exempla quædam attulisse, quibus ejusmodi reductio locum habet. Si scilicet hæc proposita sit Functio $y^5 + zzy + y^3z + \frac{z^3}{y}$; post levem attentionem apparebit, eam ad homogeneitatem perduci, posito $z = xx$: prodibit enim $y^5 + x^4y + y^3xx + \frac{x^6}{y}$, Functio homogenea 5 dimensionum ipsarum x & y. Deinde hæc Functio $y + y^2x + y^3xx + y^5x^4 + \frac{a}{x}$ ad homogeneitatem reducitur ponendo $x = \frac{1}{z}$, prodit enim Functio unius dimensionis $y + \frac{yy}{z} + \frac{y^3}{zz} + \frac{y^5}{z^4} + az$. Multo difficiliores autem sunt casus, quibus non per tam simplicem substitutionem ad homogeneitatem pervenire licet.

94. Tandem inprimis notari meretur Functionum integrarum secundum ordines divisio satis usitata, secundum quam ordo definitur ex maximo dimensionum numero qui in Functione inest. Sic $xx + yy + zz + ay - aa$ est Functio secundi ordinis, quia duæ dimensiones occurrunt. Et $y^4 + yz^3 - ay^2z + abyz -$

abyz —

$abyz - aayy + b^4$ pertinet ad Functiones quarti ordinis. Ad hanc divisionem potissimum in doctrina de lineis curvis respici solet; unde adhuc una Functionum integrarum divisio commemoranda venit. Cap. V.

95. Superest scilicet divisio Functionum integrarum in complexas atque incomplexas. Functio autem complexa est, quæ in Factores rationales resolvi potest, seu quæ est productum ex duabus Functionibus pluribusve rationalibus; cujusmodi est $y^4 - z^4 + 2az^3 - 2byzz - aazz + 2abzy - bbyy$, quæ est productum ex his duabus Functionibus $(yy + zz - az + by)$ $(yy - zz + az - by)$. Ita vidimus omnem Functionem integram homogeneam, quæ tantum duas variabiles complectatur, esse Functionem complexam, quoniam tot Factores simplices formæ $\alpha y + \beta z$ habet, quot continet dimensiones. Functio igitur integra erit incomplexa, si in Factores rationales resolvi omnino nequeat; uti $yy + zz - aa$, cujus nullos dari Factores rationales facile intelligitur. Ex inquisitione Divisorum patebit, utrum Functio proposita sit complexa an incomplexa.

CAPUT VI.

De Quantitatibus exponentialibus ac Logarithmis.

96. Quanquam notio Functionum transcendentium in calculo integrali demum perpendetur, tamen antequam eo perveniamus, quasdam species magis obvias, atque ad plures investigationes aditum aperientes, evolvere conveniet. Primum ergo considerandæ sunt quantitates exponentiales, seu Potestates, quarum Exponens ipse est quantitas variabilis. Perspicuum enim est hujusmodi quantitates ad Functiones algebraicas referri non posse, cum in his Exponentes non nisi constantes locum habeant. Multiplices autem sunt quan-

LIB. I. titates exponentiales, prout vel ſolus Exponens eſt quantitas variabilis, vel præterea etiam ipſa quantitas elevata; prioris generis eſt a^z, hujus vero y^z; quin etiam ipſe Exponens poteſt eſſe quantitas exponentialis uti in his formis a^{a^z}; a^{y^z}; y^{a^z}; x^{y^z}. Hujuſmodi autem quantitatum non plura conſtituemus genera, cum earum natura ſatis clarè intelligi queat, ſi primam tantum ſpeciem a^z evolverimus.

97. Sit igitur propoſita hujuſmodi quantitas exponentialis a^z, quæ eſt Poteſtas quantitatis conſtantis a, Exponentem habens variabilem z. Cum igitur iſte Exponens z, omnes numeros determinatos in ſe complectatur, primum patet ſi loco z omnes numeri integri affirmativi ſucceſſive ſubſtituantur, loco a^z hos prodituros eſſe valores determinatos a^1; a^2; a^3; a^4; a^5; a^6; &c. Sin autem pro z ponantur ſucceſſive numeri negativi -1, -2, -3, &c. prodibunt $\frac{1}{a}$; $\frac{1}{a^2}$; $\frac{1}{a^3}$; $\frac{1}{a^4}$; &c. ac, ſi fuerit $z = 0$, habebitur ſemper $a^0 = 1$. Quod ſi loco z numeri fracti ponantur, ut $\frac{1}{2}$; $\frac{1}{3}$; $\frac{2}{3}$; $\frac{1}{4}$; $\frac{3}{4}$; &c. orientur iſti valores $\sqrt{a}$; $\sqrt[3]{a}$; $\sqrt[3]{aa}$; $\sqrt[4]{a}$; $\sqrt[4]{a^3}$; &c., qui in ſe ſpectati geminos pluresve induunt valores, cum radicum extractio ſemper valores multiformes producat. Interim tamen hoc loco valores tantum primarii, reales ſcilicet atque affirmativi admitti ſolent; quia quantitas a^z tanquam Functio uniformis ipſius z ſpectatur. Sic $a^{\frac{5}{2}}$ medium quendam tenebit locum inter a^2 & a^3, eritque ideo quantitas ejuſdem generis; & quamvis valor $a^{\frac{5}{2}}$ ſit æque $= -aa\sqrt{a}$, ac $= +aa\sqrt{a}$, tamen poſterior tantum in cenſum venit. Eodem modo res ſe habet, ſi Exponens z valores irrationales accipiat, quibus caſibus cum difficile ſit numerum valorum involutorum concipere,

pere, unicus tantum realis consideratur. Sic $a^{\sqrt{7}}$ erit valor determinatus intra limites a^2 & a^3 comprehensus. CAP. VI.

98. Maxime autem valores quantitatis exponentialis a^z a magnitudine numeri constantis a pendebunt. Si enim fuerit $a = 1$, semper erit $a^z = 1$, quicunque valores Exponenti z tribuatur; sin autem fuerit $a > 1$, tum valor ipsius a^z eo erunt majores, quo major numerus loco z substituatur, atque adeo, posito $z = \infty$, in infinitum excrescunt; si fuerit $z = 0$, fiet $a^z = 1$, &, si sit $z < 0$ valores a^z fient unitate minores, quoad posito $z = -\infty$ fiat $a^z = 0$. Contrarium evenit si sit $a < 1$, verum tamen quantitas affirmativa; tum enim valores ipsius a^z decrescent, crescente z supra 0; crescent vero, si pro z numeri negativi substituantur. Cum enim sit $a < 1$, erit $\frac{1}{a} > 1$; posito ergo $\frac{1}{a} = b$; erit $a^z = b^{-z}$, unde posterior casus ex priori dijudicari poterit.

99. Si sit $a = 0$, ingens saltus in valoribus ipsius a^z deprehenditur, quamdiu enim fuerit z numerus affirmativus seu major nihilo, erit perpetuo $a^z = 0$: si sit $z = 0$ erit $a^0 = 1$; sin autem fuerit z numerus negativus, tum a^z obtinebit valorem infinite magnum. Sit enim $z = -3$; erit $a^z = 0^{-3} = \frac{1}{0^3} = \frac{1}{0}$, ideoque infinitum. Multo majores autem saltus occurrent, si quantitas constans a habeat valorem negativum, puta -2; tum enim ponendis loco z numeris integris valores ipsius a^z alternatim erunt affirmativi & negativi, ut ex hac Serie intelligitur

$$a^{-4};\ a^{-3};\ a^{-2};\ a^{-1};\ a^0;\ a^1;\ a^2;\ a^3;\ a^4;\ \&c.$$

+

LIB. I. $+\frac{1}{16}$; $-\frac{1}{8}$; $+\frac{1}{4}$; $-\frac{1}{2}$; 1; -2; $+4$; -8; $+16$. Præterea vero ſi Exponenti z valores tribuantur fracti, Poteſtas $a^z = (-2)^z$ mox reales mox imaginarios induet valores, erit enim $a^{\frac{1}{2}} = \sqrt{-2}$, imaginarium; at erit $a^{\frac{1}{3}} = \sqrt[3]{-2} = -\sqrt[3]{2}$ reale: utrum autem, ſi Exponenti z tribuantur valores irrationales, Poteſtas a^z exhibeat quantitates reales an imaginarias, ne quidem definiri licet.

100. His igitur incommodis numerorum negativorum loco a ſubſtituendorum commemoratis, ſtatuamus a eſſe numerum affirmativum, & unitate quidem majorem, quia huc quoque illi caſus, quibus a eſt numerus affirmativus unitate minor, facile reducuntur. Si ergo ponatur $a^z = y$, loco z ſubſtituendo omnes numeros reales, qui intra limites $+\infty$ & $-\infty$ continentur, y adipiſcetur omnes valores affirmativos intra limites $+\infty$ & 0 contentos. Si enim ſit $z = \infty$ erit $y = \infty$; ſi $z = 0$ erit $y = 1$, & ſi $z = -\infty$ fiet $y = 0$. Viciſſim ergo quicunque valor affirmativus pro y accipiatur, dabitur quoque valor realis reſpondens pro z ita ut ſit $a^z = y$; ſin autem ipſi y tribueretur valor negativus, Exponens z valorem realem habere non poterit.

101. Si igitur fuerit $y = a^z$, erit y Functio quædam ipſius z, &, quemadmodum y a z pendeat, ex natura Poteſtatum facile intelligitur; hinc enim quicunque valor ipſi z tribuatur, valor ipſius y determinatur. Erit autem $yy = a^{2z}$; $y^3 = a^{3z}$: & generaliter erit $y^n = a^{nz}$; unde ſequitur fore $\sqrt{y} = a^{\frac{1}{2}z}$; $\sqrt[3]{y} = a^{\frac{1}{3}z}$ & $\frac{1}{y} = a^{-z}$; $\frac{1}{yy} = a^{-2z}$; & $\frac{1}{\sqrt{y}} = a^{-\frac{1}{2}z}$, & ita porro. Præterea, ſi fuerit $v = a^x$ erit $vy = a^{x+z}$ & $\frac{v}{y} = a^{x-z}$, quorum ſubſidiorum beneficio eo facilius valor ipſius y ex dato valore ipſius z inveniri poteſt.

EXEM-

EXEMPLUM.

Si fuerit $a = 10$, ex Arithmetica, qua utimur, denaria in promtu erit valores ipſius y exhibere, ſi quidem pro z numeri integri ponantur. Erit enim $10^1 = 10$; $10^2 = 100$; $10^3 = 1000$; $10^4 = 10000$; & $10^0 = 1$; item $10^{-1} = \frac{1}{10} = 0,1$; $10^{-2} = \frac{1}{100} = 0,01$; $10^{-3} = \frac{1}{1000} = 0,001$: ſin autem pro z Fractiones ponantur, ope radicum extractionis valores ipſius y indicari poſſunt: ſic erit $10^{\frac{1}{2}} = \sqrt{10} = 3,162277$, &c.

102. Quemadmodum autem, dato numero a, ex quovis valore ipſius z reperiri poteſt valor ipſius y, ita viciſſim, dato valore quocunque affirmativo ipſius y, conveniens dabitur valor ipſius z, ut ſit $a^z = y$; iſte autem valor ipſius z, quatenus tanquam Functio ipſius y ſpectatur, vocari ſolet LOGARITHMUS ipſius y. Supponit ergo doctrina Logarithmorum numerum certum conſtantem loco a ſubſtituendum, qui propterea vocatur *baſis* Logarithmorum; qua aſſumta erit Logarithmus cujuſque numeri y Exponens Poteſtatis a^z, ita ut ipſa Poteſtas a^z æqualis ſit numero illi y; indicari autem Logarithmus numeri y ſolet hoc modo ly. Quod ſi ergo fuerit $a^z = y$, erit $z = ly$: ex quo intelligitur, baſin Logarithmorum, etiamſi ab arbitrio noſtro pendeat, tamen eſſe debere numerum unitate majorem: hincque nonniſi numerorum affirmativorum Logarithmos realiter exhiberi poſſe.

103. Quicunque ergo numerus pro baſi Logarithmica a accipiatur, erit ſemper $l1 = 0$; ſi enim in æquatione $a^z = y$, quæ convenit cum hac $z = ly$, ponatur $y = 1$, erit $z = 0$. Deinde numerorum unitate majorum Logarithmi erunt affirmativi, pendentes a valore baſis a, ſic erit $la = 1$; $laa = 2$; $la^3 = 3$; $la^4 = 4$,

LIB. I. $la^4 = 4$; &c., unde a poſteriori intelligi poteſt, quantus numerus pro baſi ſit aſſumtus, ſcilicet ille numerus, cujus Logarithmus eſt $= 1$, erit baſis Logarithmica. Numerorum autem unitate minorum, affirmativorum tamen, Logarithmi erunt negativi; erit enim $l\frac{1}{a} = -1$; $l\frac{1}{aa} = -2$; $l\frac{1}{a^3} = -3$, &c.; numerorum autem negativorum Logarithmi non erunt reales, ſed imaginarii, uti jam notavimus.

104. Simili modo ſi fuerit $ly = z$; erit $lyy = 2z$; $ly^3 = 3z$; & generaliter $ly^n = nz$, ſeu $ly^n = nly$, ob $z = ly$. Logarithmus igitur cujuſque Poteſtatis ipſius y æquatur Logarithmo ipſius y per Exponentem Poteſtatis multiplicato; ſic erit $l\sqrt{y} = \frac{1}{2}z = \frac{1}{2}ly$; $l\frac{1}{\sqrt{y}} = ly^{-\frac{1}{2}} = -\frac{1}{2}ly$; & ita porro; unde ex dato Logarithmo cujuſque numeri inveniri poſſunt Logarithmi quarumcunque ipſius Poteſtatum. Sin autem jam inventi ſint duo Logarithmi, nempe $ly = z$ & $lv = x$: cum ſit $y = a^z$ & $v = a^x$ erit $lvy = x + z = lv + ly$; hinc Logarithmus Producti duorum numerorum æquatur ſummæ Logarithmorum Factorum; ſimili vero modo erit $l\frac{y}{v} = z - x = ly - lv$; hincque Logarithmus Fractionis æquatur Logarithmo Numeratoris dempto Logarithmo Denominatoris, quæ regulæ inſerviunt Logarithmis plurium numerorum inveniendis, ex cognitis jam aliquot Logarithmis.

105. Ex his autem patet aliorum numerorum non dari Logarithmos rationales, niſi Poteſtatum baſeos a; niſi enim numerus alius b fuerit Poteſtas baſis a, ejus Logarithmus numero rationali exprimi non poterit. Neque vero etiam Logarithmus ipſius b erit numerus irrationalis; ſi enim foret $lb = \sqrt{n}$, tum eſſet $a^{\sqrt{n}} = b$; id quod fieri nequit, ſi quidem numeri a & b rationales ſtatuantur; ſolent autem imprimis numerorum rationa-

tionalium & integrorum Logarithmi deſiderari, quia ex his Logarithmi Fractionum ac numerorum ſurdorum inveniri poſſunt. Cum igitur Logarithmi numerorum, qui non ſunt Poteſtates baſis a, neque rationaliter neque irrationaliter exhiberi queant, merito ad quantitates tranſcendentes referuntur, hincque Logarithmi quantitatibus transcendentibus annumerari ſolent.

106. Hanc ob rem Logarithmi numerorum vero tantum proxime per Fractiones decimales exprimi ſolent, qui eo minus à veritate diſcrepabunt, ad quo plures figuras fuerint exacti. Atque hoc modo per ſolam radicis quadratæ extractionem cujuſque numeri Logarithmus vero proxime determinari poterit. Cum enim, poſito $ly = z$ & $lv = x$, ſit $l\sqrt{vy} = \frac{x+z}{2}$; ſi numerus propoſitus b contineatur intra limites a^2 & a^3, quorum Logarithmi ſunt 2 & 3, quæratur valor ipſius $a^{2\frac{1}{2}}$ ſeu $a^2\sqrt{a}$, atque b vel intra limites a^2 & $a^{2\frac{1}{2}}$ vel $a^{2\frac{1}{2}}$ & a^3 continebitur, utrumvis accidat, ſumendo medio proportionali, denuo limites propiores prodibunt, hocque modo ad limites pervenire licebit, quorum intervallum data quantitate minus evadat, & quibuſcum numerus propoſitus b ſine errore confundi poſſit. Quoniam vero horum ſingulorum limitum Logarithmi dantur, tandem Logarithmus numeri b reperietur.

EXEMPLUM.

Ponatur baſis Logarithmica $a = 10$, quod in tabulis uſu receptis fieri ſolet; & quæratur vero tantum proxime Logarithmus numeri 5; quia hic continetur intra limites 1 & 10 quorum Logarithmi ſunt 0 & 1; ſequenti modo radicum extractio continua inſtituatur, quoad ad limites à numero propoſito 5 non amplius diſcrepantes perveniatur.

$A = 1,000000;$	$lA = 0,0000000$	sit
$B = 10,000000;$	$lB = 1,0000000;$	$C = \sqrt{AB}$
$C = 3,162277;$	$lC = 0,5000000;$	$D = \sqrt{BC}$
$D = 5,623413;$	$lD = 0,7500000;$	$E = \sqrt{CD}$
$E = 4,216964;$	$lE = 0,6250000;$	$F = \sqrt{DE}$
$F = 4,869674;$	$lF = 0,6875000;$	$G = \sqrt{DF}$
$G = 5,232991;$	$lG = 0,7187500;$	$H = \sqrt{FG}$
$H = 5,048065;$	$lH = 0,7031250;$	$I = \sqrt{FH}$
$I = 4,958069;$	$lI = 0,6953125;$	$K = \sqrt{HI}$
$K = 5,002865;$	$lK = 0,6992187;$	$L = \sqrt{IK}$
$L = 4,980416;$	$lL = 0,6972656;$	$M = \sqrt{KL}$
$M = 4,991627;$	$lM = 0,6982421;$	$N = \sqrt{KM}$
$N = 4,997242;$	$lN = 0,6987304;$	$O = \sqrt{KN}$
$O = 5,000052;$	$lO = 0,6989745;$	$P = \sqrt{NO}$
$P = 4,998647;$	$lP = 0,6988525;$	$Q = \sqrt{OP}$
$Q = 4,999350;$	$lQ = 0,6989135;$	$R = \sqrt{OQ}$
$R = 4,999701;$	$lR = 0,6989440;$	$S = \sqrt{OR}$
$S = 4,999876;$	$lS = 0,6989592;$	$T = \sqrt{OS}$
$T = 4,999963;$	$lT = 0,6989668;$	$V = \sqrt{OT}$
$V = 5,000008;$	$lV = 0,6989707;$	$W = \sqrt{TV}$
$W = 4,999984;$	$lW = 0,6989687;$	$X = \sqrt{WV}$
$X = 4,999997;$	$lX = 0,6989697;$	$Y = \sqrt{VX}$
$Y = 5,000003;$	$lY = 0,6989702;$	$Z = \sqrt{XY}$
$Z = 5,000000;$	$lZ = 0,6989700;$	

Sic ergo mediis proportionalibus ſumendis tandem perventum eſt ad $Z = 5,000000$, ex quo Logarithmus numeri 5 quæſitus eſt $0,698970$, poſita baſi Logarithmica $= 10$. Quare erit proxime $10^{\frac{69897}{100000}} = 5$. Hoc autem modo computatus eſt canon Logarithmorum vulgaris à Briggio & Vlacquio, quamquam poſtea eximia inventa ſunt compendia, quorum ope multo expeditius Logarithmi ſupputari poſſunt.

107. Dantur ergo tot diverſa Logarithmorum ſyſtemata quot varii numeri pro baſi a accipi poſſunt, atque ideo numerus ſyſtema-

tematum Logarithmicorum erit infinitus. Perpetuo autem in duobus ſyſtematis Logarithmi ejuſdem numeri eandem inter ſe ſervant rationem. Sit baſis unius ſyſtematis $= a$, alterius $= b$, atque numeri n Logarithmus in priori ſyſtemate $= p$, in poſteriori $= q$; erit $a^{p} = n$ & $b^{q} = n$; unde $a^{p} = b^{q}$; ideoque $a = b^{\frac{q}{p}}$. Oportet ergo ut Fractio $\frac{q}{p}$ conſtantem obtineat valorem, quicunque numerus pro n fuerit aſſumtus. Quod ſi ergo pro uno ſyſtemate Logarithmi omnium numerorum fuerint computati, hinc facili negotio per regulam auream Logarithmi pro quovis alio ſyſtemate reperiri poſſunt. Sic, cum dentur Logarithmi pro baſi 10, hinc Logarithmi pro quavis alia baſi, puta 2, inveniri poſſunt; quæratur enim Logarithmus numeri n pro baſi 2, qui ſit $= q$, cum ejuſdem numeri n Logarithmus ſit $= p$ pro baſi 10. Quoniam pro baſi 10 eſt $l2 = 0,3010300$, & pro baſi 2, eſt $l2 = 1$, erit $0,3010300 : 1 = p : q$ ideoque $q = \frac{p}{0,3010300} = 3,3219277 \cdot p$, ſi ergo omnes Logarithmi communes multiplicentur per numerum $3,3219277$, prodibit tabula Logarithmorum pro baſi 2.

108. *Hinc ſequitur duorum numerorum Logarithmos in quocunque ſyſtemate eandem tenere rationem*

Sint enim duo numeri M & N, quorum pro baſi a Logarithmi ſint m & n, erit $M = a^{m}$ & $N = a^{n}$: hinc fiet $^{mn} = M^{n} = N^{m}$, ideoque $M = N^{\frac{m}{n}}$; in qua æquatione cum baſis a non amplius inſit, perſpicuum eſt Fractionem $\frac{m}{n}$ habere valorem à baſi a non pendentem. Sint enim pro alia baſi b numerorum eorundem M & N Logarithmi μ & ν, pari modo colligetur fore $M = N^{\frac{\mu}{\nu}}$. Erit ergo $N^{\frac{m}{n}} = N^{\frac{\mu}{\nu}}$, hincque $\frac{m}{n} = \frac{\mu}{\nu}$, ſeu $m : n = \mu : \nu$. Ita jam vidimus

 in omni Logarithmorum ſyſtemate Logarithmos diverſarum ejuſdem numeri Poteſtatum ut y^m & y^n tenere rationem Exponentium $m : n$.

109. Ad canonem ergo Logarithmorum pro baſi quacunque a condendum ſufficit numerorum tantum primorum Logarithmos methodo ante tradita, vel alia commodiori, ſupputaſſe. Cum enim Logarithmi numerorum compoſitorum ſint æquales ſummis Logarithmorum ſingulorum Factorum, Logarithmi numerorum compoſitorum per ſolam additionem reperientur. Sic, ſi habeantur Logarithmi numerorum 3 & 5, erit $l\,15 = l\,3 + l\,5$; $l\,45 = 2\,l\,3 + l\,5$. Atque, cum ſupra pro baſi $a = 10$, inventus ſit $l\,5 = 0,\,6989700$, præterea autem ſit $l\,10 = 1$ erit $l\,\frac{10}{5} = l\,2 = l\,10 - l\,5$, ideoque orietur $l\,2 = 1 - 0,\,6989700 = 0,\,3010300$; ex his autem numerorum primorum 2 & 5 Logarithmis inventis reperientur Logarithmi omnium numerorum ex his 2 & 5 compoſitorum; cujuſmodi ſunt iſti 4, 8, 16, 32, 64, &c; 20, 40, 80, 25, 50; &c.

110. Tabularum autem Logarithmicarum ampliſſimus eſt uſus in contrahendis calculis numericis, propterea quod ex ejuſmodi tabulis non ſolum dati cujuſque numeri Logarithmus, ſed etiam cujuſque Logarithmi propoſiti numerus conveniens reperiri poteſt. Sic, ſi c, d, e, f, g, h, denotent numeros quoſcunque, citra multiplicationem reperiri poterit valor iſtius expreſſionis $\frac{ccd\sqrt{e}}{f\sqrt[3]{gh}}$, erit enim hujus expreſſionis Logarithmus $= 2\,lc + ld + \frac{1}{2}\,le - lf - \frac{1}{3}\,lg - \frac{1}{3}\,lh$, cui Logarithmo ſi quæratur numerus reſpondens, habebitur valor quæſitus. Inprimis autem inſerviunt tabulæ Logarithmicæ dignitatibus atque radicibus intricatiſſimis inveniendis, quarum operationum loco in Logarithmis tantum multiplicatio & diviſio adhibetur.

Exem-

EXEMPLUM I.

Quæratur valor hujus Poteſtatis $2^{\frac{7}{12}}$: quoniam ejus Logarithmus eſt $= \frac{7}{12} l\, 2$, multiplicetur Logarithmus binarii ex tabulis qui eſt 0, 3010300 per $\frac{7}{12}$ hoc eſt per $\frac{1}{2} + \frac{1}{12}$ erit, $l\, 2^{\frac{7}{12}} = 0, 1756008$, cui Logarithmo reſpondet numerus 1, 498307, qui ergo proxime exhibet valorem $2^{\frac{7}{12}}$.

EXEMPLUM II.

Si numerus incolarum cujuſpiam provinciæ quotannis ſui parte trigeſima augeatur, initio autem in provincia habitaverint 100000 hominum, quæritur poſt 100 annos incolarum numerus. Sit brevitatis gratia initio incolarum numerus $= n$, ita ut ſit $n = 100000$, anno elapſo uno erit incolarum numerus $= (1 + \frac{1}{30})n = \frac{31}{30} n$: poſt duos annos $= (\frac{31}{30})^2 n$: poſt tres annos $= (\frac{31}{30})^3 n$, hincque poſt centum annos $= (\frac{31}{30})^{100} n = (\frac{31}{30})^{100} 100000$; cujus Logarithmus eſt $= 100\, l \frac{31}{30} + l\, 100000$. At eſt $l \frac{31}{30} = l\, 31 - l\, 30 = 0, 014240439$, unde $100\, l \frac{31}{30} = 1, 4240439$, ad quem ſi addatur $l\, 100000 = 5$, erit Logarithmus numeri incolarum quæſiti $= 6\, 4240439$, cui reſpondet numerus $= 2654874$. Poſt centum ergo annos numerus incolarum ſit plus quam vicies ſexies cum ſemiſſe major.

EXEM-

LIB. I.

EXEMPLUM III.

Cum poſt diluvium à ſex hominibus genus humanum ſit propagatum, ſi ponamus ducentis annis poſt, numerum hominum jam ad 1000000 excreviſſe, quæritur quanta ſui parte numerus hominum quotannis augeri debuerit. Ponamus hoc tempore numerum hominum parte ſua $\frac{1}{x}$ quotannis increviſſe, atque poſt ducentos annos prodierit neceſſe eſt numerus hominum $= \left(\frac{1+x}{x}\right)^{200} 6 = 1000000$, unde fit $\frac{1+x}{x} = \left(\frac{1000000}{6}\right)^{\frac{1}{200}}$. Erit ergo $l\frac{1+x}{x} = \frac{1}{200}\ l\frac{1000000}{6} = \frac{1}{200}$. $5,2218487 = 0,0261092$, ideoque $\frac{1+x}{x} = \frac{1061963}{1000000}$, & $1000000 = 61963\,x$, unde fit $x = 16$ circiter. Ad tantam ergo hominum multiplicationem ſuffeciſſet, ſi quotannis decima ſexta ſui parte increverint; quæ multiplicatio ob longævam vitam non nimis magna cenſeri poteſt. Quod ſi autem eadem ratione per intervallum 400 annorum numerus hominum creſcere perrexiſſet, tum numerus hominum ad $1000000 . \frac{1000000}{6} = 166666666666$ aſcendere debuiſſet, quibus ſuſtentandis univerſus orbis terrarum nequaquam par fuiſſet.

EXEMPLUM IV.

Si ſingulis ſeculis numerus hominum duplicetur, quæritur incrementum annuum. Si quotannis hominum numerum parte ſua $\frac{1}{x}$ creſcere ponamus, & initio numerus hominum fuerit $= n$, erit is poſt centum annos $= \left(\frac{1+x}{x}\right)^{100} n$, qui cum eſſe debeat

beat $= 2n$, erit $\frac{1+x}{x} = 2^{\frac{1}{100}}$ & $l\,\frac{1+x}{x} = \frac{1}{100}\,l2 =$ 0, 0030103; hinc $\frac{1+x}{x} = \frac{10069555}{10000000}$; ergo $x = \frac{10000000}{69555} = 144$, circiter; ſufficit ergo ſi numerus hominum quotannis parte ſua $\frac{1}{144}$ augeatur. Quam ob cauſam maxime ridiculæ ſunt eorum incredulorum hominum objectiones, qui negant tam brevi temporis ſpatio ab uno homine univerſam terram incolis impleri potuiſſe.

111. Potiſſimum autem Logarithmorum uſus requiritur ad ejuſmodi æquationes reſolvendas, in quibus quantitas incognita in Exponentem ingreditur. Sic, ſi ad hujuſmodi perveniatur æquationem $a^x = b$, ex qua incognitæ x valorem erui oporteat, hoc non niſi per Logarithmos effici poterit. Cum enim ſit $a^x = b$ erit $l\,a^x = x\,l\,a = l\,b =$ ideoque $x = \frac{l\,b}{l\,a}$, ubi quidem perinde eſt, quonam ſyſtemate Logarithmico utatur, cum in omni ſyſtemate Logarithmi numerorum a & b eandem inter ſe teneant rationem.

Exemplum I.

Si numerus hominum quotannis centeſima ſui parte augeatur; quæritur poſt quot annos numerus hominum fiat decuplo major. Ponamus hoc evenire poſt x annos, & initio hominum numerum fuiſſe $= n$, erit is ergo elapſis x annis $= (\frac{101}{100})^x n$, qui cum æqualis ſit $10n$, fiet $(\frac{101}{100})^x = 10$; ideoque $x\,l\,\frac{101}{100} = l\,10$ & $x = \frac{l\,10}{l\,101 - l\,100}$. Prodibit itaque $x = \frac{10000000}{43214} = 231$. Poſt annos ergo 231 fiet homi-

LIB. I. num numerus, quorum incrementum annuum tantum centesimam partem efficit, decuplo major; hinc post 462 annos fiet centies, & post 693 annos millies major.

EXEMPLUM. II.

Quidam debet 400000 florenos hac conditione ut quotannis usuram 5 de centenis solvere teneatur; exsolvit autem singulis annis 25000 florenos: quæritur post quot annos debitum penitus extinguatur. Scribamus a pro debita summa 400000 fl. & b pro summa 25000 fl. quotannis soluta; debebit ergo elapso uno anno $\frac{105}{100}a - b$; elapsis duobus annis $(\frac{105}{100})^2 a - \frac{105}{100}b - b$; elapsis tribus annis $(\frac{105}{100})^3 a - (\frac{105}{100})^2 b - \frac{105}{100}b - b$; hinc, posito brevitatis causa, n pro $\frac{105}{100}$, elapsis x annis adhuc debebit $n^x a - n^{x-1} b - n^{x-2} b - n^{x-3} b - \ldots\ldots\ldots - b = n^x a - b(1 + n + n^2 + \ldots\ldots + n^{x-1})$. Cum igitur sit ex natura progressionum geometricarum, $1 + n + n^2 + \ldots\ldots\ldots + n^{x-1} = \frac{n^x - 1}{n - 1}$, post x annos debitor adhuc debebit $n^x a - \frac{n^x b - b}{n - 1}$ flor., quod debitum nihilo æquale positum dabit hanc æquationem $n^x a = \frac{n^x b - b}{n - 1}$, seu $(n - 1) n^x a = n^x b - b$, ideoque $(b - na + a) n^x = b$ & $n^x = \frac{b}{b - (n - 1) a}$, unde fit $x = \frac{lb - l(b - (n - 1) a)}{ln}$.

Cum jam sit $a = 400000$, $b = 25000$, $n = \frac{105}{100}$, erit $(n - 1) a = 20000$ & $b - (n - 1) a = 5000$, atque annorum, quibus debitum penitus extinguitur, numerus $x =$

l

$\frac{l\,25000 - l\,5000}{l\,\frac{105}{100}} = \frac{l\,5}{l\,\frac{21}{20}} = \frac{6989700}{211893}$; erit ergo x aliquanto minor quam 33; ſcilicet elapſis annis 33 non ſolum debitum extinguetur, ſed creditor debitori reddere tenebitur $\frac{(n^{33}-1)b}{n-1}$ Cap. VI.

$$-n^{33}a = \frac{(\frac{21}{20})^{33} \cdot 5000 - 25000}{\frac{1}{20}} = 100000\left(\frac{21}{20}\right)^{33} - 500000$$

flor. Quia vero eſt $l\,\frac{21}{20} = 0{,}0211892991$, erit $l\left(\frac{21}{20}\right)^{33} = 0{,}69924687$, & $l\,100000\left(\frac{21}{20}\right)^{33} = 5{,}6992469$, cui reſpondet hic numerus 500318, 8; unde creditor debitori poſt 33 annos reſtituere debet $318\,\frac{4}{5}$ florenos.

112. Logarithmi autem vulgares ſuper baſi $= 10$ extracti, præter hunc uſum, quem Logarithmi in genere præſtant, in Arithmetica decimali uſu recepta ſingulari gaudent commodo, atque ob hanc cauſam præ aliis ſyſtematibus inſignem afferunt utilitatem. Cum enim Logarithmi omnium numerorum, præter denarii Poteſtates, in Fractionibus decimalibus exhibeantur, numerorum inter 1 & 10 contentorum Logarithmi intra limites 0 & 1, numerorum autem inter 10 & 100 contentorum Logarithmi inter limites 1 & 2, & ita porro, continebuntur. Conſtat ergo Logarithmus quiſque ex numero integro & Fractione decimali; & ille numerus integer vocari ſolet CHARACTERISTICA; Fractio decimalis autem MANTISSA. Characteriſtica itaque unitate deficiet a numero notarum, quibus numerus conſtat; ita Logarithmi numeri 78509 Characteriſtica erit 4, quia is ex quinque notis ſeu figuris conſtat. Hinc ex Logarithmo cujuſvis numeri ſtatim intelligitur, ex quot figuris numerus ſit compoſitus. Sic numerus Logarithmo 7, 5804631 reſpondens ex 8 figuris conſtabit.

113. Si ergo duorum Logarithmorum Mantiſſæ conveniant, Characteriſticæ vero tantum diſcrepent, tum numeri his Logarithmis

LIB. I. rithmis respondentes rationem habebunt, ut Potestas denarii ad unitatem, ideoque ratione figurarum, quibus constant, convenient. Ita horum Logarithmorum 4, 9130187 & 6 9130187 numeri erunt 81850 & 8185000; Logarithmo autem 3 9130187 conveniet 8185, & Logarithmo huic 0, 9130187 convenit 8, 185. Sola ergo Mantissa indicabit figuras numerum componentes, quibus inventis, ex Characteristica patebit, quot figuræ a sinistra ad integra referri debeant, reliquæ ad dextram vero dabunt Fractiones decimales. Sic, si hic Logarithmus fuerit inventus 2, 7603429, Mantissa indicabit has figuras 5758945, Characteristica 2 autem numerum illi Logarithmo determinat, ut sit 575, 8945; si Characteristica esset 0, foret numerus 5, 758945; sin denuo unitate minuatur ut sit — 1, erit numerus respondens decies minor, nempe 0, 5758945; & Characteristicæ — 2 respondebit 0, 05758945 &c.: loco Characteristicarum autem hujusmodi negativarum — 1, — 2, — 3, &c. scribi solent 9, 8, 7, &c., atque subintelligitur hos Logarithmos denario minui debere. Hæc vero in manductionibus ad tabulas Logarithmorum fusius exponi solent.

EXEMPLUM.

Si hæc progressio 2, 4, 16, 256, &c., *cujus quisque terminus est quadratum præcedentis, continuetur usque ad terminum vigesimum quintum; quæritur magnitudo hujus termini ultimi.* Termini hujus progressionis per Exponentes ita commodius exprimuntur: 2^1, 2^2, 2^4, 2^8, &c. ubi patet Exponentes progressionem geometricam constituere, atque termini vigesimi quinti exponentem fore $2^{24} = 16777216$, ita ut ipse terminus quæsitus sit $= 2^{16777216}$, hujus ergo Logarithmus erit $= 16777216.\,l2$. Cum ergo sit $l2 = 0,30102999566398 1195$, erit numeri quæsiti Logarithmus $= 5050445,25973367$, ex cujus Characteristica patet numerum quæsitum more solito expressum constare ex 5050446 figuris. Mantissa autem 25973367593 2 in tabula

la Logarithmorum quæsita dabit figuras initiales numeri quæsiti, quæ erunt 181858. Quanquam ergo iste numerus nullo modo exhiberi potest, tamen affirmari potest eum omnino ex 5050446 figuris constare, atque figuras initiales sex esse 181858, quas dextrorsum adhuc 5050440 figuræ sequantur, quarum insuper nonnullæ ex majori Logarithmorum canone definiri possent, undecim scilicet figuræ initiales erunt 18185852986. Cap. VI.

CAPUT VII.

De quantitatum exponentialium ac Logarithmorum per Series explicatione.

114. Quia est $a^0 = 1$, atque crescente Exponente ipsius a simul valor Potestatis augetur, si quidem a est numerus unitate major; sequitur si Exponens infinite parum cyphram excedat, Potestatem ipsam quoque infinite parum unitatem esse superaturam. Sit ω numerus infinite parvus, seu Fractio tam exigua, ut tantum non nihilo sit æqualis, erit $a^\omega = 1 + \psi$, existente ψ quoque numero infinite parvo. Ex præcedente enim capite constat nisi ψ esset numerus infinite parvus, neque ω talem esse posse. Erit ergo vel $\psi = \omega$, vel $\psi > \omega$, vel $\psi < \omega$, quæ ratio utique a quantitate litteræ a pendebit, quæ cum adhuc sit incognita, ponatur $\psi = k\,\omega$, ita ut sit $a^\omega = 1 + k\,\omega$; &, sumta a pro basi Logarithmica, erit $\omega = l(1 + k\,\omega)$.

EXEMPLUM.

Quo clarius appareat, quemadmodum numerus k pendeat a basi a, ponamus esse $a = 10$; atque ex tabulis vulgaribus quæramus Logarithmum numeri quam minime unitatem superantis,

LIB. I. rantis, puta $1 + \frac{1}{1000000}$, ita ut ſit $k\omega = \frac{1}{1000000}$; erit $l(1 + \frac{1}{1000000}) = l\frac{1000001}{1000000} = 0{,}00000043429 = \omega$. Hinc, ob $k\omega = 0{,}00000100000$, erit $\frac{1}{k} = \frac{43429}{100000}$ & $k = \frac{100000}{43429} = 2{,}30258$: unde patet k eſſe numerum finitum pendentem a valore baſis a. Si enim alius numerus pro baſi a ſtatuatur, tum Logarithmus ejuſdem numeri $1 + k\omega$ ad priorem datam tenebit rationem, unde ſimul alius valor litteræ k prodiret.

115. Cum ſit $a^{\omega} = 1 + k\omega$, erit $a^{i\omega} = (1 + k\omega)^{i}$, quicunque numerus loco i ſubſtituatur. Erit ergo $a^{i\omega} = 1 + \frac{i}{1}k\omega + \frac{i(i-1)}{1.\ 2}k^2\omega^2 + \frac{i(i-1)(i-2)}{1.\ 2.\ 3}k^3\omega^3 + \&c.$ Quod ſi ergo ſtatuatur $i = \frac{z}{\omega}$, & z denotet numerum quemcunque finitum, ob ω numerum infinite parvum, fiet i numerus infinite magnus, hincque $\omega = \frac{z}{i}$, ita ut ſit ω Fractio denominatorem habens infinitum, adeoque infinite parva, qualis eſt aſſumta. Subſtituatur ergo $\frac{z}{i}$ loco ω, eritque $a^{z} = (1 + \frac{kz}{i})^{i} = 1 + \frac{1}{1}kz + \frac{1(i-1)}{1.\ 2i}k^2z^2 + \frac{1(i-1)(i-2)}{1.\ 2i.\ 3i}k^3z^3 + \frac{1(i-1)(i-2)(i-3)}{1.\ 2i.\ 3i.\ 4i.}k^4z^4 + \&c.$, quæ æquatio erit vera ſi pro i numerus infinite magnus ſubſtituatur. Tum vero eſt k numerus definitus ab a pendens, uti modo vidimus.

116. Cum autem i ſit numerus infinite magnus, erit $\frac{i-1}{i} = 1$; patet enim quo major numerus loco i ſubſtituatur, eo propius valorem Fractionis $\frac{i-1}{i}$ ad unitatem eſſe acceſſurum, hinc ſi

i ſit

i ſit numerus omni aſſignabili major, Fractio quoque $\frac{i-1}{i}$ ipſam unitatem adæquabit. Ob ſimilem autem rationem erit $\frac{i-2}{i} = 1$; $\frac{i-3}{i} = 1$; & ita porro; hinc ſequitur fore $\frac{i-1}{2i} = \frac{1}{2}$; $\frac{i-2}{3i} = \frac{1}{3}$; $\frac{i-3}{4i} = \frac{1}{4}$; & ita porro. His igitur valoribus ſubſtitutis, erit $a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1.2} + \frac{k^3z^3}{1.2.3} + \frac{k^4z^4}{1.2.3.4} +$ &c. in infinitum. Hæc autem æquatio ſimul relationem inter numeros a & k oſtendit, poſito enim $z = 1$, erit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \frac{k^4}{1.2.3.4} +$ &c., atque ut a ſit $= 10$, neceſſe eſt ut ſit circiter $k = 2,30258$, uti ante invenimus.

117. Ponamus eſſe $b = a^n$, erit, ſumto numero a pro baſi Logarithmica, $lb = n$. Hinc, cum ſit $b^z = a^{nz}$, erit per Seriem infinitam $b^z = 1 + \frac{knz}{1} + \frac{k^2n^2z^2}{1.2} + \frac{k^3n^3z^3}{1.2.3} + \frac{k^4n^4z^4}{1.2.3.4.} +$ &c., poſito vero lb pro n, erit $b^z = 1 + \frac{kz}{1}lb + \frac{k^2z^2}{1.2}(lb)^2 + \frac{k^3z^3}{1.2.3}(lb)^3 + \frac{k^4z^4}{1.2.3.4}(lb)^4 +$ &c.. Cognito ergo valore litteræ k ex dato valore baſis a, quantitas exponentialis quæcunque b^z per Seriem infinitam exprimi poterit, cujus termini ſecundum Poteſtates ipſius z procedant. His expoſitis oſtendamus quoque quomodo Logarithmi per Series infinitas explicari poſſint.

118. Cum ſit $a^\omega = 1 + k\omega$, exiſtente ω Fractione infinite parva, atque ratio inter a & k definiatur per hanc æquationem $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} +$ &c., ſi a ſumatur pro baſi Logarithmica, erit $\omega = l(1 + k\omega)$ & $i\omega = l(1 + k\omega)^i$.

Mani-

LIB. I. Manifeſtum autem eſt, quo major numerus pro i ſumatur, eo magis Poteſtatem $(1+k\omega)^i$ unitatem eſſe ſuperaturam; atque ſtatuendo $i =$ numero infinito, valorem Poteſtatis $(1+k\omega)^i$ ad quemvis numerum unitate majorem aſcendere. Quod ſi ergo ponatur $(1+k\omega)^i = 1+x$, erit $l(1+x) = i\omega$, unde, cum ſit $i\omega$ numerus finitus, Logarithmus ſcilicet numeri $1+x$, perſpicuum eſt, i eſſe debere numerum infinite magnum, alioquin enim $i\omega$ valorem finitum habere non poſſet.

119. Cum autem poſitum ſit $(1+k\omega)^i = 1+x$, erit $1+k\omega = (1+x)^{\frac{1}{i}}$ & $k\omega = (1+x)^{\frac{1}{i}} - 1$, unde fit $i\omega = \frac{i}{k}((1+x)^{\frac{1}{i}} - 1)$. Quia vero eſt $i\omega = l(1+x)$, erit $l(1+x) = \frac{i}{k}(1+x)^{\frac{1}{i}} - \frac{i}{k}$, poſito i numero infinite magno. Eſt autem $(1+x)^{\frac{1}{i}} = 1 + \frac{1}{i}x - \frac{1(i-1)}{i.\ 2i}x^2 + \frac{1(i-1)(2i-1)}{i.\ 2i.\ 3i}x^3 - \frac{1(i-1)(2i-1)(3i-1)}{i.\ 2i.\ 3i.\ 4i}x^4 + \&c.$ Ob i autem numerum infinitum, erit $\frac{i-1}{2i} = \frac{1}{2}$; $\frac{2i-1}{3i} = \frac{2}{3}$; $\frac{3i-1}{4i} = \frac{3}{4}$, &c.; hinc erit $i(1+x)^{\frac{1}{i}} = i + \frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$, & conſequenter $l(1+x) = \frac{1}{k}(\frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.)$, poſita baſi Logarithmica $= a$ ac denotante k numerum huic baſi convenientem, ut ſcilicet ſit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \&c.$

120. Cum igitur habeamus Seriem Logarithmo numeri $1+x$ æqualem, ejus ope ex data baſi a definire poterimus valorem numeri

numeri k. Si enim ponamus $1+x=a$, ob $la=1$, erit $1=\frac{1}{k}\left(\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\&c.\right)$, hincque habebitur $k=\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\&c.$, cujus ideo Seriei infinitæ valor, ſi ponatur $a=10$, circiter eſſe debebit $=2,30258$; quanquam difficulter intelligi poteſt eſſe $2,30258=\frac{9}{1}-\frac{9^2}{2}+\frac{9^3}{3}-\frac{9^4}{4}+\&c.$, quoniam hujus Seriei termini continuo fiunt majores, neque adeo aliquot terminis ſumendis ſumma vero propinqua haberi poteſt: cui incommodo mox remedium afferetur. Cap. VII.

121. Quoniam igitur eſt $l(1+x)=\frac{1}{k}\left(\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\&c.\right)$, erit, poſito x negativo, $l(1-x)=-\frac{1}{k}\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\&c.\right)$. Subtrahatur Series poſterior a priori, erit $l(1+x)-l(1-x)=l\frac{1+x}{1-x}=\frac{2}{k}\times\left(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\&c.\right)$. Nunc ponatur $\frac{1+x}{1-x}=a$, ut ſit $x=\frac{a-1}{a+1}$, ob $la=1$ erit $k=2\left(\frac{a-1}{a+1}+\frac{(a-1)^3}{3(a+1)^3}+\frac{(a-1)^5}{5(a+1)^5}+\&c.\right)$, ex qua æquatione valor numeri k ex baſi a inveniri poterit. Si ergo baſis a ponatur $=10$ erit $k=2\left(\frac{9}{11}+\frac{9^3}{3.11^3}+\frac{9^5}{5.11^5}+\frac{9^7}{7.11^7}+\&c.\right)$, cujus Seriei termini ſenſibiliter decreſcunt, ideoque mox valorem pro k ſatis propinquum exhibent.

122. Quoniam ad ſyſtema Logarithmorum condendum baſin a pro lubitu accipere licet, ea ita aſſumi poterit ut fiat $k=1$. Ponamus ergo eſſe $k=1$, eritque per Seriem ſupra

(116) inventam, $a = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} +$ &c., qui termini, si in fractiones decimales convertantur atque actu addantur, præbebunt hunc valorem pro $a =$ 2,71828182845904523536028, cujus ultima adhuc nota veritati est consentanea. Quod si jam ex hac basi Logarithmi construantur, ii vocari solent Logarithmi *naturales* seu *hyperbolici*, quoniam quadratura hyperbolæ per istiusmodi Logarithmos exprimi potest. Ponamus autem brevitatis gratia pro numero hoc 2,718281828459 &c. constanter litteram e, quæ ergo denotabit basin Logarithmorum naturalium seu hyperbolicorum, cui respondet valor litteræ $k = 1$; sive hæc littera e quoque exprimet summam hujus Seriei $1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} +$ &c. in infinitum.

123. Logarithmi ergo hyperbolici hanc habebunt proprietatem, ut numeri $1 + \omega$ Logarithmus sit $= \omega$, denotante ω quantitatem infinite parvam, atque cum ex hac proprietate valor $k = 1$ innotescat, omnium numerorum Logarithmi hyperbolici exhiberi poterunt. Erit ergo, posita e pro numero supra invento, perpetuo $e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} +$ &c. ipsi vero Logarithmi hyperbolici ex his Seriebus invenientur, quibus est $l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} +$ &c, & $l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} +$ &c., quæ Series vehementer convergunt, si pro x statuatur fractio valde parva: ita ex Serie posteriori facili negotio inveniuntur Logarithmi numerorum unitate non multo majorum. Posito namque $x = \frac{1}{5}$, erit $l\frac{6}{4} = l\frac{3}{2} = \frac{2}{1.5} + \frac{2}{3.5^3} + \frac{2}{5.5^5} + \frac{2}{7.5^7} +$ &c., & facto $x = \frac{1}{7}$, erit $l\frac{4}{3} = \frac{2}{1.7} + \frac{2}{3.7^3} + \frac{2}{5.7^5} +$

$\frac{2}{7.7}$

$\frac{2}{7.7^7}$ + &c., facto $x = \frac{1}{9}$, erit $l\frac{5}{4} = \frac{2}{1.9} + \frac{2}{3.9^3} + \frac{2}{5.9^5} + \frac{2}{7.9^7}$ + &c.. Ex Logarithmis vero harum fractionum reperientur Logarithmi numerorum integrorum, erit enim ex natura Logarithmorum $l\frac{3}{2} + l\frac{4}{3} = l2$; tum $l\frac{3}{2} + l2 = l3$; & $2l2 = l4$; porro $l\frac{5}{4} + l4 = l5$; $l2 + l3 = l6$; $3l2 = l8$; $2l3 = l9$; & $l2 + l5 = l10$. Cap. VII.

Exemplum.

Hinc Logarithmi hyperbolici numerorum ab 1 usque ad 10 ita se habebunt, ut sit

$l1$	=	0,	00000	00000	00000	00000	00000
$l2$	=	0,	69314	71805	59945	30941	72321
$l3$	=	1,	09861	22886	68109	69139	52452
$l4$	=	1,	38629	43611	19890	61883	44642
$l5$	=	1,	60943	79124	34100	37460	07593
$l6$	=	1,	79175	94692	28055	00081	24773
$l7$	=	1,	94591	01490	55313	30510	54639
$l8$	=	2,	07944	15416	79835	92825	16964
$l9$	=	2,	19722	45773	36219	38279	04905
$l10$	=	2,	30258	50929	94045	68401	79914

Hi scilicet Logarithmi omnes ex superioribus tribus Seriebus sunt deducti, præter $l7$, quem hoc compendio sum assecutus. Posui nimirum in Serie posteriori $x = \frac{1}{99}$ sicque obtinui $l\frac{100}{98} = l\frac{50}{49} = 0{,}02020270731751944840782 30$, qui subtractus a $l50 = 2l5 + l2 = 3{,}91202300542814605861875 08$, relinquit $l49$, cujus semissis dat $l7$.

LIB. I.

124. Ponatur Logarithmus hyperbolicus ipſius $1+x$ ſeu $l(1+x)=y$; erit $y=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+$&c.. Sumto autem numero a pro baſi Logarithmica, ſit numeri ejuſdem $1+x$ Logarithmus $=v$; erit, ut vidimus, $v=\frac{1}{k}(x-\frac{xx}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c.)=\frac{y}{k}$; hincque $k=\frac{y}{v}$; ex quo commodiſſime valor ipſius k baſi a reſpondens ita definitur ut ſit æqualis cujuſvis numeri Logarithmo hyperbolico diviſo per Logarithmum ejuſdem numeri ex baſi a formati. Poſito ergo numero hoc $=a$, erit $v=1$, hincque fit $k=$ Logarithmo hyperbolico baſis a. In ſyſtemate ergo Logarithmorum communium, ubi eſt $a=10$, erit $k=$ Logarithmo hyperbolico ipſius 10, unde fit $k=2,302585092994045684017991 4$, quem valorem jam ſupra ſatis prope collegimus. Si ergo ſinguli Logarithmi hyperbolici per hunc numerum k dividantur, vel, quod eodem redit, multiplicentur per hanc fractionem decimalem $0,43429448190325182765112 89$, prodibunt Logarithmi vulgares baſi $a=10$ convenientes.

125. Cum ſit $e^z=1+\frac{z}{1}+\frac{z^2}{1.2}+\frac{z^3}{1.2.3}+$&c., ſi ponatur $a^y=e^z$, erit, ſumtis Logarithmis hyperbolicis, $yla=z$, quia eſt $le=1$, quo valore loco z ſubſtituto, erit $a^y=1+\frac{yla}{1}+\frac{y^2(la)^2}{1.2}+\frac{y^3(la)^3}{1.2.3}+$&c., unde quælibet quantitas exponentialis ope Logarithmorum hyperbolicorum per Seriem infinitam explicari poteſt. Tum vero, denotante i numerum infinite magnum, tam quantitates exponentiales quam Logarithmi per poteſtates exponi poſſunt. Erit enim $e^z=(1+\frac{z}{i})^i$, hincque $a^y=(1+\frac{yla}{i})^i$, deinde pro Logarithmis hyperbolicis habetur $l(1+x)=i((1+x)^{\frac{1}{i}}-1)$. De cetero

tero Logarithmorum hyperbolicorum uſus in calculo integrali fuſius demonſtrabitur. Cap. VII.

CAPUT VIII.

De quantitatibus tranſcendentibus ex Circulo ortis.

126. Poſt Logarithmos & quantitates exponentiales conſiderari debent Arcus circulares eorumque Sinus & Coſinus, quia non ſolum aliud quantitatum tranſcendentium genus conſtituunt, ſed etiam ex ipſis Logarithmis & exponentialibus, quando imaginariis quantitatibus involvuntur, proveniunt, id quod infra clarius patebit.

Ponamus ergo Radium Circuli ſeu Sinum totum eſſe $= 1$, atque ſatis liquet Peripheriam hujus Circuli in numeris rationalibus exacte exprimi non poſſe, per approximationes autem inventa eſt Semicircumferentia hujus Circuli eſſe $=$ 3, 14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651327230664709384 46 +, pro quo numero, brevitatis ergo, ſcribam π, ita ut ſit $\pi =$ Semicircumferentiæ Circuli, cujus Radius $= 1$, ſeu π erit longitudo Arcus 180 graduum.

127. Denotante z Arcum hujus Circuli quemcunque, cujus Radium perpetuo aſſumo $= 1$; hujus Arcus z conſiderari potiſſimum ſolent Sinus & Coſinus. Sinum autem Arcus z in poſterum hoc modo indicabo, *ſin. A.* z, ſeu tantum *ſin.* z. Coſinum vero hoc modo *coſ. A.* z, ſeu tantum *coſ.* z. Ita, cum π ſit Arcus 180°, erit $\textit{ſin.}\, 0\pi = 0$; $\textit{coſ.}\, 0\pi = 1$; &

$\textit{ſin.}\, \frac{1}{2}\pi = 1$, $\textit{coſ.}\, \frac{1}{2}\pi = 0$; $\textit{ſin.}\, \pi = 0$; $\textit{coſ.}\, \pi = -1$;

$\textit{ſin.}\, \frac{3}{2}\pi = -1$; $\textit{coſ.}\, \frac{3}{2}\pi = 0$; $\textit{ſin.}\, 2\pi = 0$; & $\textit{coſ.}\, 2\pi = 1$.

Omnes ergo Sinus & Coſinus intra limites $+1$ & -1 con-

LIB. I. tinentur. Erit autem porro $cos. z = sin. (\frac{1}{2}\pi - z)$, & $sin. z = cos. (\frac{1}{2}\pi - z)$, atque $(sin. z)^2 + (cos. z)^2 = 1$. Præter has denominationes notandæ ſunt quoque hæ: *tang.* z, quæ denotat Tangentem Arcus z; *cot.* z Cotangentem Arcus z; conſtatque eſſe $tang. z = \frac{sin. z}{cos. z}$ & $cot. z = \frac{cos. z}{sin. z} = \frac{1}{tang. z}$; quæ omnia ex Trigonometria ſunt nota.

128. Hinc vero etiam conſtat ſi habeantur duo Arcus y & z, fore $sin. (y + z) = sin. y. cos. z + cos. y. sin. z$, & $cos. (y + z) = cos. y. cos. z - sin. y. sin. z$, itemque $sin. (y - z) = sin. y. cos. z - cos. y. sin. z$ & $cos. (y - z) = cos. y. cos. z + sin. y. sin. z$.

Hinc loco y ſubſtituendo Arcus $\frac{1}{2}\pi$; π; $\frac{3}{2}\pi$, &c., erit

$sin. (\frac{1}{2}\pi + z) = + cos. z$	$sin. (\frac{1}{2}\pi - z) = + cos. z$
$cos. (\frac{1}{2}\pi + z) = - sin. z$	$cos. (\frac{1}{2}\pi - z) = + sin. z$
$sin. (\pi + z) = - sin. z$	$sin. (\pi - z) = + sin. z$
$cos. (\pi + z) = - cos. z$	$cos. (\pi - z) = - cos. z$
$sin. (\frac{3}{2}\pi + z) = - cos. z$	$sin. (\frac{3}{2}\pi - z) = - cos. z$
$cos. (\frac{3}{2}\pi + z) = + sin. z$	$cos. (\frac{3}{2}\pi - z) = - sin. z$
$sin. (2\pi + z) = + sin. z$	$sin. (2\pi - z) = - sin. z$
$cos. (2\pi + z) = + cos. z$	$cos. (2\pi - z) = + cos. z$

Si

Si ergo n denotet numerum integrum quemcunque, erit

CAP. VIII.

$\sin.\left(\frac{4n+1}{2}\pi+z\right)=+\cos.z$	$\sin.\left(\frac{4n+1}{2}\pi-z\right)=+\cos.z$
$\cos.\left(\frac{4n+1}{2}\pi+z\right)=-\sin.z$	$\cos.\left(\frac{4n+1}{2}\pi-z\right)=-\sin.z$
$\sin.\left(\frac{4n+2}{2}\pi+z\right)=-\sin.z$	$\sin.\left(\frac{4n+2}{2}\pi-z\right)=+\sin.z$
$\cos.\left(\frac{4n+2}{2}\pi+z\right)=-\cos.z$	$\cos.\left(\frac{4n+2}{2}\pi-z\right)=-\cos.z$
$\sin.\left(\frac{4n+3}{2}\pi+z\right)=-\cos.z$	$\sin.\left(\frac{4n+3}{2}\pi-z\right)=-\cos.z$
$\cos.\left(\frac{4n+3}{2}\pi+z\right)=+\sin.z$	$\cos.\left(\frac{4n+3}{2}\pi-z\right)=-\sin.z$
$\sin.\left(\frac{4n+4}{2}\pi+z\right)=+\sin.z$	$\sin.\left(\frac{4n+4}{2}\pi-z\right)=-\sin.z$
$\cos.\left(\frac{4n+4}{2}\pi+z\right)=+\cos.z$	$\cos.\left(\frac{4n+4}{2}\pi-z\right)=+\cos.z$

Quæ formulæ veræ ſunt ſive n ſit numerus affirmativus ſive negativus integer.

129. Sit $\sin.z=p$ & $\cos.z=q$ erit $pp+qq=1$; & $\sin.y=m$; $\cos.y=n$; ut ſit quoque $mm+nn=1$; Arcuum ex his compoſitorum Sinus & Coſinus ita ſe habebunt.

$\sin.\ z=p$	$\cos.\ z=q$
$\sin.(y+z)=mq+np$	$\cos.(y+z)=nq-mp$
$\sin.(2y+z)=2mnq+(nn-mm)p$	$\cos.(2y+z)=(nn-mm)q-2mnp$
$\sin.(3y+z)=(3mn^2-m^3)q+(n^3-3m^2n)p$	$\cos.(3y+z)=(n^3-3m^2n)q-(3mn^2-m^3)p$
&c.	&c.

Arcus iſti z, $y+z$, $2y+z$, $3y+z$, &c., in arithmetica progreſſione progrediuntur; eorum vero tam Sinus quam Coſinus progreſſionem recurrentem conſtituunt, qualis ex denominatore $1-2nx+(mm+nn)xx$ oritur; eſt enim

ſin.

LIB. I. $\sin.(2y+z) = 2n\sin.(y+z) - (mm+nn)\sin.z$ five
$\sin.(2y+z) = 2\cos.y.\sin.(y+z) - (\sin.z)$; atque fimili modo
$\cos.(2y+z) = 2\cos.y.\cos.(y+z) - \cos.z$. Eodem modo erit porro
$\sin.(3y+z) = 2\cos.y.\sin.(2y+z) - \sin.(y+z)$, &
$\cos.(3y+z) = 2\cos.y.\cos.(2y+z) - \cos.(y+z)$, itemque
$\sin.(4y+z) = 2\cos.y.\sin.(3y+z) - \sin.(2y+z)$, &
$\cos.(4y+z) = 2\cos.y.\cos.(3y+z) - \cos.(2y+z)$ &c.

Cujus legis beneficio Arcuum in progreffione arithmetica progredientium tam Sinus quam Cofinus quoufque libuerit expedite formari poffunt.

130. Cum fit $\sin.(y+z) = \sin.y.\cos.z + \cos.y\sin.z$ atque $\sin.(y-z) = \sin.y.\cos.z - \cos.y.\sin.z$, erit his expreffionibus vel addendis vel fubtrahendis:

$$\sin.y.\cos.z = \frac{\sin.(y+z) + \sin.(y-z)}{2}$$

$$\cos.y.\sin.z = \frac{\sin.(y+z) - \sin.(y-z)}{2}$$

Quia porro eft $\cos.(y+z) = \cos.y.\cos.z - \sin.y.\sin.z$, atque $\cos.(y-z) = \cos.y.\cos.z + \sin.y.\sin.z$, erit pari modo

$$\cos.y.\cos.z = \frac{\cos.(y-z) + \cos.(y+z)}{2}$$

$$\sin.y.\sin.z = \frac{\cos.(y-z) - \cos.(y+z)}{2}.$$

Sit $y = z = \frac{1}{2}v$, erit ex his poftremis formulis:

$$(\cos.\tfrac{1}{2}v)^2 = \frac{1+\cos.v}{2}, \ \& \ \cos.\tfrac{1}{2}v = \sqrt{\frac{1+\cos.v}{2}}$$

$$(\sin.\tfrac{1}{2}v)^2 = \frac{1-\cos.v}{2}, \ \& \ \sin.\tfrac{1}{2}v = \sqrt{\frac{1-\cos.v}{2}}$$

unde, ex dato Cofinu cujufque anguli reperiuntur ejus femiffis Sinus & Cofinus.

131. Ponatur Arcus $y+z = a$, & $y-z = b$; erit $y = \frac{a+b}{2}$ & $z = \frac{a-b}{2}$, quibus in fuperioribus formulis fubftitutis

tutis, habebuntur hæ æquationes, tanquam totidem Theoremata. CAP. VIII.

$$\text{ſin.}\,a + \text{ſin.}\,b = 2\,\text{ſin.}\,\frac{a+b}{2}\,\text{coſ.}\,\frac{a-b}{2}$$

$$\text{ſin.}\,a - \text{ſin.}\,b = 2\,\text{coſ.}\,\frac{a+b}{2}\,\text{ſin.}\,\frac{a-b}{2}$$

$$\text{coſ.}\,a + \text{coſ.}\,b = 2\,\text{coſ.}\,\frac{a+b}{2}\,\text{coſ.}\,\frac{a-b}{2}$$

$$\text{coſ.}\,b - \text{coſ.}\,a = 2\,\text{ſin.}\,\frac{a+b}{2}\,\text{ſin.}\,\frac{a-b}{2}$$

ex his porro naſcuntur, ope diviſionis, hæc Theoremata

$$\frac{\text{ſin.}\,a + \text{ſin.}\,b}{\text{ſin.}\,a - \text{ſin.}\,b} = \text{tang.}\,\frac{a+b}{2}.\ \text{cot.}\,\frac{a-b}{2} = \frac{\text{tang.}\,\frac{a+b}{2}}{\text{tang.}\,\frac{a-b}{2}}$$

$$\frac{\text{ſin.}\,a + \text{ſin.}\,b}{\text{coſ.}\,a + \text{coſ.}\,b} = \text{tang.}\,\frac{a+b}{2}$$

$$\frac{\text{ſin.}\,a + \text{ſin.}\,a}{\text{coſ.}\,b - \text{coſ.}\,a} = \text{cot.}\,\frac{a-b}{2}$$

$$\frac{\text{ſin.}\,a - \text{ſin.}\,b}{\text{coſ.}\,a + \text{coſ.}\,b} = \text{tang.}\,\frac{a-b}{2}$$

$$\frac{\text{ſin.}\,a - \text{ſin.}\,a}{\text{coſ.}\,b - \text{coſ.}\,a} = \text{cot.}\,\frac{a+b}{2}$$

$$\frac{\text{coſ.}\,a + \text{coſ.}\,b}{\text{coſ.}\,b - \text{coſ.}\,a} = \text{cot.}\,\frac{a+b}{2}.\ \text{cot.}\,\frac{a-b}{2}$$

Ex his denique deducuntur iſta Theoremata

$$\frac{\text{ſin.}\,a + \text{ſin.}\,b}{\text{coſ.}\,a + \text{coſ.}\,b} = \frac{\text{coſ.}\,b - \text{coſ.}\,a}{\text{ſin.}\,a - \text{ſin.}\,b};$$

$$\frac{\text{ſin.}\,a + \text{ſin.}\,b}{\text{ſin.}\,a - \text{ſin.}\,b} \times \frac{\text{coſ.}\,a + \text{coſ.}\,b}{\text{coſ.}\,b - \text{coſ.}\,a} = \left(\text{cot.}\,\frac{a-b}{2}\right)^2$$

$$\frac{\text{ſin.}\,a + \text{ſin.}\,b}{\text{ſin.}\,a - \text{ſin.}\,b} \times \frac{\text{coſ.}\,b - \text{coſ.}\,a}{\text{coſ.}\,a + \text{coſ.}\,b} = \left(\text{tang.}\,\frac{a+b}{2}\right)^2$$

132. Cum ſit $(\text{ſin.}\,z)^2 + (\text{coſ.}\,z)^2 = 1$ erit, Factoribus ſumendis, $(\text{coſ.}\,z + \sqrt{-1}.\,\text{ſin.}\,z)(\text{coſ.}\,z - \sqrt{-1}.\,\text{ſin.}\,z) = 1$; qui Factores, etſi imaginarii, tamen ingentem præſtant uſum in Arcubus combinandis & multiplicandis. Quæratur enim productum horum Factorum $(\text{coſ.}\,z + \sqrt{-1}.\,\text{ſin.}\,z)(\text{coſ.}\,y + \sqrt{-1}.\,\text{ſin.}\,y)$ ac reperietur $\text{coſ.}\,y.\,\text{coſ.}\,z - \text{ſin.}\,y.\,\text{ſin.}\,z + (\text{coſ.}\,y.\,\text{ſin.}\,z + \text{ſin.}\,y.\,\text{coſ.}\,z)$

LIB. I. $\sqrt{-1}$. Cum autem sit $cos. y. cos. z - sin. y. sin. z = cos. (y+z)$ & $cos. y. sin. z + sin. y. cos. z = sin. (y+z)$ erit hoc productum $(cos. y. + \sqrt{-1}. sin. y.)(cos. z + \sqrt{-1}. sin. z) = cos. (y+z) + \sqrt{-1}. sin. (y+z)$

& simili modo

$(cos. y - \sqrt{-1}. sin. y)(cos. z - \sqrt{-1}. sin. z) = cos. (y+z) - \sqrt{-1}. sin. (y+z)$

item

$(cos. x \pm \sqrt{-1}. sin. x)(cos. y \pm \sqrt{-1}. sin. y)(cos. z \pm \sqrt{-1}. sin\ z) = cos. (x+y+z) \pm \sqrt{-1}. sin. (x+y+z)$.

133. Hinc itaque sequitur fore $(cos. z \pm \sqrt{-1}. sin\ z)^2 = cos. 2z \pm \sqrt{-1}. sin. 2z$, & $(cos. z \pm \sqrt{-1}. sin. z)^3 = cos. 3z \pm \sqrt{-1}. sin. 3z$.

ideoque generaliter erit $(cos. z \pm \sqrt{-1}. sin. z)^n = cos. nz \pm \sqrt{-1}. sin. nz$:

Unde, ob signorum ambiguitatem, erit

$$cos. nz = \left(\frac{(cos\ z + \sqrt{-1}. sin. z)^n + (cos. z - \sqrt{-1}. sin. z)^n}{2}\right) \ \&$$

$$sin.\ nz = \frac{(cos. z + \sqrt{-1}. sin. z)^n - (cos. z - \sqrt{-1}. sin. z)^n}{2\sqrt{-1}}$$

Evolutis ergo binomiis hisce erit per Series:

$$cos. nz = (cos. z)^n - \frac{n(n-1)}{1.2}(cos. z)^{n-2}(sin. z)^2 + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}(cos. z)^{n-4}(sin. z)^4 - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.3.4.5.6}(cos. z)^{n-6}(sin. z)^6 + \&c., \ \&$$

$$sin. nz = \frac{n}{1}(cos. z)^{n-1} sin. z - \frac{n(n-1)(n-2)}{1.2.3}(cos. z)^{n-3}(sin. z)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}(cos. z)^{n-5}(sin. z)^5 - \&c.$$

134.

134. Sit Arcus z infinite parvus, erit $\mathit{ſin.}\ z = z$ & $\mathit{coſ.}\ z = 1$: ſit autem n numerus infinite magnus, ut ſit Arcus nz finitæ magnitudinis, puta, $nz = v$; ob $\mathit{ſin.}\ z = z = \frac{v}{n}$ erit CAP. VIII.

$$\mathit{coſ.}\ v = 1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3.4.5.6} + \&c.,\ \&$$

$$\mathit{ſin.}\ v = v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3.4.5.6.7} + \&c.$$ Dato ergo Arcu v, ope harum Serierum ejus Sinus & Coſinus inveniri poterunt; quarum formularum uſus quo magis pateat, ponamus Arcum v eſſe ad quadrantem, ſeu $90°$, ut m ad n, ſeu eſſe $v = \frac{m}{n} . \frac{\pi}{2}$; Quia nunc valor ipſius π conſtat, ſi is ubique ſubſtituatur, prodibit

$$\mathit{ſin.}\ A.\ \frac{m}{n}\ 90° =$$

$$+ \frac{m}{n} . 1,5707963267948966192313216916$$
$$- \frac{m^3}{n^3} . 0,6459640975062462536557565636$$
$$+ \frac{m^5}{n^5} . 0,0796926262461670451205055488$$
$$- \frac{m^7}{n^7} . 0,0046817541353186881006854632$$
$$+ \frac{m^9}{n^9} . 0,0001604411847873598218726605$$
$$- \frac{m^{11}}{n^{11}} . 0,0000035988432352120853404580$$
$$+ \frac{m^{13}}{n^{13}} . 0,0000000569217292196792681171$$
$$- \frac{m^{15}}{n^{15}} . 0,0000000006688035109811467224$$
$$+ \frac{m^{17}}{n^{17}} . 0,0000000000060669357311061950$$

$- \frac{m^{19}}{n^{19}} \cdot 0,0000000000000437706546731370$

$+ \frac{m^{21}}{n^{21}} \cdot 0,0000000000000002571422892856$

$- \frac{m^{23}}{n^{23}} \cdot 0,0000000000000000012538995403$

$+ \frac{m^{25}}{n^{25}} \cdot 0,0000000000000000000051564550$

$- \frac{m^{27}}{n^{27}} \cdot 0,0000000000000000000000181239$

$+ \frac{m^{29}}{n^{29}} \cdot 0,0000000000000000000000000549$

atque *cos.* A. $\frac{m}{n}\, 90^{\circ} =$

$+ \; 1,0000000000000000000000000000$

$- \frac{m^{2}}{n^{2}} \cdot 1,2337005501361698273543113745$

$+ \frac{m^{4}}{n^{4}} \cdot 0,2536695079010480136365633659$

$- \frac{m^{6}}{n^{6}} \cdot 0,0208634807633529608730516364$

$+ \frac{m^{8}}{n^{8}} \cdot 0,0009192602748394265802417158$

$- \frac{m^{10}}{n^{10}} \cdot 0,0000252020423730606054810526$

$+ \frac{m^{12}}{n^{12}} \cdot 0,0000004710874778818171503665$

$- \frac{m^{14}}{n^{14}} \cdot 0,0000000063866030837918522408$

$+ \frac{m^{16}}{n^{16}} \cdot 0,0000000000656596311497947230$

$- \frac{m^{18}}{n^{18}} \cdot 0,0000000000005294400200734620$

$+ \frac{m^{20}}{n^{20}} \cdot 0,0000000000000034377391790981$

$- \frac{m^{22}}{n^{22}} \cdot 0,0000000000000000183599165212$

+

$$+ \frac{m^{24}}{n^{24}} \cdot 0,0000000000000000000820675327$$

$$- \frac{m^{26}}{n^{26}} \cdot 0,0000000000000000000003115285$$

$$+ \frac{m^{28}}{n^{28}} \cdot 0,0000000000000000000000010165$$

$$- \frac{m^{30}}{n^{30}} \cdot 0,0000000000000000000000000026$$

Cum igitur ſufficiat Sinus & Coſinus angulorum ad 45° noſſe, fractio $\frac{m}{n}$ ſemper minor erit quam $\frac{1}{2}$, hincque etiam ob Poteſtates fractionis $\frac{m}{n}$, Series exhibitæ maxime convergent, ita ut plerumque aliquot tantum termini ſufficiant, præcipue, ſi Sinus & Coſinus non ad tot figuras deſiderentur.

135. Inventis Sinibus & Coſinibus inveniri quidem poſſunt Tangentes & Cotangentes, per analogias conſuetas, at quia in hujuſmodi ingentibus numeris multiplicatio & diviſio vehementer eſt moleſta, peculiari modo eas exprimere convenit. Erit ergo

$$tang.\, v = \frac{ſin.\, v}{coſ.\, v} = \frac{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2\,3....7} + \&c.}{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.0....6} + \&c.}$$

$$\& \; cot.\, v = \frac{coſ.\, v}{ſin.\, v} = \frac{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3....6} + \&c.}{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2\,3....7} + \&c.}$$

LIB. I. ſi jam ſit Arcus $v = \frac{m}{n}\,90^\circ$ erit eodem modo quo ante

$tang.\ A.\frac{m}{n}\,90^\circ =$	$cot.\ A.\frac{m}{n}\,90^\circ =$
$+\frac{2mn}{nn-mm}\cdot 0{,}6366197723675$	$+\frac{n}{m}\ 0{,}6366197723675$
$+\frac{m}{n}\cdot 0{,}2975567820597$	$\frac{-4mn}{4nn-mm}\cdot 0{,}3183098861837$
$+\frac{m^3}{n^3}\cdot 0{,}0186886502773$	$-\frac{m}{n}\cdot 0{,}2052888894145$
$+\frac{m^5}{n^5}\cdot 0{,}0018424752034$	$-\frac{m^3}{n^3}\cdot 0{,}0065510747882$
$+\frac{m^7}{n^7}\cdot 0{,}0001975800714$	$-\frac{m^5}{n^5}\cdot 0{,}0003450292554$
$+\frac{m^9}{n^9}\cdot 0{,}0000216977245$	$-\frac{m^7}{n^7}\cdot 0{,}0000202791060$
$+\frac{m^{11}}{n^{11}}\cdot 0{,}0000024011370$	$-\frac{m^9}{n^9}\cdot 0{,}0000012366527$
$+\frac{m^{13}}{n^{13}}\cdot 0{,}0000002664132$	$-\frac{m^{11}}{n^{11}}\cdot 0{,}0000000764959$
$+\frac{m^{15}}{n^{15}}\cdot 0{,}0000000295864$	$-\frac{m^{13}}{n^{13}}\cdot 0{,}0000000047597$
$+\frac{m^{17}}{n^{17}}\cdot 0{,}0000000032867$	$-\frac{m^{15}}{n^{15}}\cdot 0{,}0000000002969$
$+\frac{m^{19}}{n^{19}}\cdot 0{,}0000000003651$	$-\frac{m^{17}}{n^{17}}\cdot 0{,}0000000000185$
$+\frac{m^{21}}{n^{21}}\cdot 0{,}0000000000405$	$-\frac{m^{19}}{n^{19}}\cdot 0{,}0000000000011$
$+\frac{m^{23}}{n^{23}}\cdot 0{,}0000000000045$	
$+\frac{m^{25}}{n^{25}}\cdot 0{,}0000000000005$	

quarum Serierum ratio infra fuſius exponetur.

136. Ex ſuperioribus quidem conſtat, ſi cogniti fuerint omnium angulorum ſemirecto minorum Sinus & Coſinus, inde ſimul omnium angulorum majorum Sinus & Coſinus haberi. Verum ſi tantum angulorum 30° minorum habeantur Sinus

Sinus & Cosinus, ex iis, per solam additionem & subtractionem, omnium angulorum majorum Sinus & Cosinus inveniri possunt. Cum enim sit $sin.\ 30° = \frac{1}{2}$, erit, posito $y = 30°$ ex (130) $cos. z = sin. (30 + z) + sin. (30 - z)$; & $sin. z = cos. (30 - z) - cos. (30 + z)$, ideoque ex Sinibus & Cosinibus angulorum z & $30 - z$, reperiuntur $sin. (30 + z) = cos. z - sin. (30 - z)$ & $cos. (30 + z) = cos. (30 - z) - sin. z$, unde Sinus & Cosinus angulorum a 30° ad 60°, hincque omnes majores definiuntur. CAP. VIII.

137. In Tangentibus & Cotangentibus simile subsidium usu venit. Cum enim sit $tang. (a + b) = \frac{tang. a + tang. b}{1 - tang. a. tang. b}$, erit $tang. 2a = \frac{2\ tang. a}{1 - tang. a. tang. a}$, & $cot. 2a = \frac{cot. a - tang. a}{2}$ unde ex Tangentibus & Cotangentibus Arcuum 30° minorum inveniuntur Cotangentes usque ad 60°.

Sit jam $a = 30 - b$ erit $2a = 60 - 2b$ & $cot. 2a = tang. (30 + 2b)$; erit ergo $tang. (30 + 2b) = \frac{cot. (30 - b) - tang. (30 - b)}{2}$, unde etiam Tangentes Arcuum 30° majorum obtinentur.

Secantes autem & Cosecantes ex Tangentibus per solam subtractionem inveniuntur; est enim $cosec. z = cot. \frac{1}{2} z - cot. z$, & hinc $sec. z = cot. (45° - \frac{1}{2} z) - tang. z$. Ex his ergo luculenter perspicitur, quomodo canones Sinuum construi potuerint.

138. Ponatur denuo in formulis §. 133, Arcus z infinite parvus, & sit n numerus infinite magnus i, ut iz obtineat valorem finitum v. Erit ergo $nz = v$; & $z = \frac{v}{i}$, unde $sin. z = \frac{v}{i}$ & $cos. z = 1$; his substitutis fit $cos. v =$

(1 +

LIB. I.

$$\frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^{i}+\left(1-\frac{v\sqrt{-1}}{i}\right)^{i}}{2}$$; atque $\mathit{fin.}\, v =$

$$\frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^{i}-\left(1-\frac{v\sqrt{-1}}{i}\right)^{i}}{2\sqrt{-1}}.$$ In Capite autem præcedente vidimus esse $\left(1+\frac{z}{i}\right)^{i}=e^{z}$, denotante e basin Logarithmorum hyperbolicorum: scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit $\mathit{cof.}\, v =$

$$\frac{e^{+v\sqrt{-1}}+e^{-v\sqrt{-1}}}{2} \;\&\; \mathit{fin.}\, v=\frac{e^{+v\sqrt{-1}}-e^{-v\sqrt{-1}}}{2\sqrt{-1}}.$$

Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Cosinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}}=\mathit{cof.}\, v+\sqrt{-1}.\mathit{fin.}\, v$ & $e^{-v\sqrt{-1}}=\mathit{cof.}\, v-\sqrt{-1}.\mathit{fin.}\, v$.

139. Sit jam in iisdem formulis §. 130. n numerus infinite parvus, seu $n=\frac{1}{i}$, existente i numero infinite magno, erit $\mathit{cof.}\, nz=\mathit{cof.}\,\frac{z}{i}=1$ & $\mathit{fin.}\, nz=\mathit{fin.}\,\frac{z}{i}=\frac{z}{i}$; Arcus enim evanescentis $\frac{z}{i}$ Sinus est ipsi æqualis, Cosinus vero $=1$. His positis habebitur

$$1=\frac{(\mathit{cof.}\, z+\sqrt{-1}.\mathit{fin.}\, z)^{\frac{1}{i}}+(\mathit{cof.}\, z-\sqrt{-1}.\mathit{fin.}\, z)^{\frac{1}{i}}}{2}$$ &

$$\frac{z}{i}=\frac{(\mathit{cof.}\, z+\sqrt{-1}.\mathit{fin.}\, z)^{\frac{1}{i}}-(\mathit{cof.}\, z-\sqrt{-1}.\mathit{fin.}\, z)^{\frac{1}{i}}}{2\sqrt{-1}}.$$ Sumendis autem Logarithmis hyperbolicis supra (125) ostendimus esse $l(1+x)=i(1+x)^{\frac{1}{i}}-i$, seu $y^{\frac{1}{i}}=1+\frac{1}{i}ly$,

posito

posito y loco $1+x$. Nunc igitur, posito loco y, partim $cos. z + \sqrt{-1}.\, sin.\, z$ partim $cos.\, z - \sqrt{-1}.\, sin.\, z$, prodibit $1 =$ CAP. VIII.

$$\frac{1+\frac{1}{i}l(cos.z+\sqrt{-1}.sin.z)+1+\frac{1}{i}l(cos.z-\sqrt{-1}.sin.z)}{2}$$

$=1$, ob Logarithmos evanescentes, ita ut hinc nil sequatur. Altera vero æquatio pro Sinu suppeditat:

$$\frac{z}{i}=\frac{\frac{1}{i}l(cos.z+\sqrt{-1}.sin.z)-\frac{1}{i}l(cos.z-\sqrt{-1}.sin.z)}{2\sqrt{-1}}$$

ideoque $z \frac{1}{2\sqrt{-1}}\, l\frac{cos.\,z+\sqrt{-1}.\,sin.\,z}{cos.\,z-\sqrt{-1}.\,sin.\,z}$, unde patet quemadmodum Logarithmi imaginarii ad Arcus circulares revocentur.

140. Cum sit $\frac{sin.z}{cos.z} = tang.\,z$, Arcus z per suam Tangentem ita exprimetur ut sit $z = \frac{1}{2\sqrt{-1}}\, l\frac{1+\sqrt{-1}.\,tang.z}{1-\sqrt{-1}.\,tang.z}$. Supra vero (§. 123) vidimus esse $l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} +$ &c.. Posito ergo $x = \sqrt{-1}.\, tang.z$, fiet $z = \frac{tang.\,z}{1} - \frac{(tang.\,z)^3}{3} + \frac{(tang.\,z)^5}{5} - \frac{(tang.\,z)^7}{7} +$ &c.. Si ergo ponamus $tang.\,z = t$, ut sit z Arcus, cujus Tangens est t, quem ita indicabimus *A. tang. t*, ideoque erit $z = A\, tang.\, t$. Cognita ergo Tangente t erit Arcus respondens $z = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} -$ &c.. Cum igitur, si Tangens t æquetur Radio 1, fiat Arcus $z =$ Arcui 45° seu $z = \frac{\pi}{4}$, erit $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c., quæ est Series a LEIBNITZIO primum producta, ad valorem Peripheriæ Circuli exprimendum.

141. Quo autem ex hujusmodi Serie longitudo Arcus Cir-

LIB. I. culi expedite definiri possit, perspicuum est pro Tangente t fractionem satis parvam substitui debere. Sic ope hujus Seriei facile reperietur longitudo Arcus z, cujus Tangens t æquetur $\frac{1}{10}$, foret enim iste Arcus $z = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} - \&c.$, cujus Seriei valor per approximationem non difficulter in fractione decimali exhiberetur. At vero ex tali Arcu cognito nihil pro longitudine totius Circumferentiæ concludere licebit, cum ratio, quam Arcus, cujus Tangens est $= \frac{1}{10}$, ad totam Peripheriam tenet, non sit assignabilis. Hanc ob rem ad Peripheriam indagandam, ejusmodi Arcus quæri debet, qui sit simul pars aliquota Peripheriæ, & cujus Tangens satis exigua commode exprimi queat. Ad hoc ergo institutum sumi solet Arcus 30°. cujus Tangens est $= \frac{1}{\sqrt{3}}$, quia minorum Arcuum cum Peripheria commensurabilium Tangentes nimis fiunt irrationales. Quare, ob Arcum $30° = \frac{\pi}{6}$, erit $\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 3^2 \sqrt{3}} - \&c.$, & $\pi = \frac{2\sqrt{3}}{1} - \frac{2\sqrt{3}}{3 \cdot 3} + \frac{2\sqrt{3}}{5 \cdot 3^2} - \frac{2\sqrt{3}}{7 \cdot 3^3} + \&c.$, cujus Seriei ope valor ipsius π ante exhibitus incredibili labore fuit determinatus.

142. Hic autem labor eo major est, quod primum singuli termini sint irrationales, tum vero quisque tantum, circiter, triplo sit minor quam præcedens. Huic itaque incommodo ita occurri poterit: sumatur Arcus 45° seu $\frac{\pi}{4}$ cujus valor, etsi per Seriem vix convergentem $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$, exprimitur, tamen is retineatur, atque in duos Arcus a & b dispertiatur ut sit $a + b = \frac{\pi}{4} = 45°$. Cum igitur sit $tang.\,(a+b) = 1 = \frac{tang.\,a + tang.\,b}{1 - tang.\,a.\,tang.\,b}$ erit $1 - tang.a.\,tang.b =$

tang. a :

$tang.\, a + tang.\, b$ & $tang.\, b = \frac{1 - tang.\, a}{1 + tang.\, a}$. Sit nunc $tang.\, a = \frac{1}{2}$, erit $tang.\, b = \frac{1}{3}$, hinc uterque Arcus a & b per Seriem rationalem multo magis, quam ſuperior, convergentem exprimetur, eorumque ſumma dabit valorem Arcus $\frac{\pi}{4}$; hinc itaque erit

CAP. VIII.

$$\pi = 4 \cdot \left\{ \begin{array}{l} \frac{1}{1.2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \frac{1}{7.2^7} + \frac{1}{9.2^9} - \&c. \\ \frac{1}{1.3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \frac{1}{7.3^7} + \frac{1}{9.3^9} - \&c. \end{array} \right\}$$

hoc ergo modo multo expeditius longitudo ſemicircumferentiæ π inveniri potuiſſet, quam quidem factum eſt ope Seriei ante commemoratæ.

C A P U T IX.

De inveſtigatione Factorum trinomialium.

143. Quemadmodum Factores ſimplices cujuſque Functionis integræ inveniri oporteat, ſupra quidem oſtendimus hoc fieri per reſolutionem æquationum. Si enim propoſita ſit Functio quæcunque integra $\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \&c.$, hujuſque quærantur Factores ſimplices formæ $p - qz$, manifeſtum eſt, ſi $p - qz$ fuerit Factor Functionis $\alpha + \beta z + \gamma z^2 + \&c.$, tum, poſito $z = \frac{p}{q}$, quo caſu Factor $p - qz$ fit $= 0$, etiam ipſam Functionem propoſitam evaneſcere debere. Hinc $p - qz$ erit Factor vel diviſor Functionis $\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \&c.$, ſequitur fore hanc

LIB. I. expressionem $\alpha + \frac{\beta p}{q} + \frac{\gamma p^2}{q^2} + \frac{\delta p^3}{q^3} + \frac{\epsilon p^4}{q^4} + \&c. = 0$. Unde vicissim, si omnes radices $\frac{p}{q}$ hujus æquationis eruantur, singulæ dabunt totidem Factores simplices Functionis integræ propositæ $\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c.$, nempe $p - qz$. Patet autem simul numerum Factorum hujusmodi simplicium ex maxima Potestate ipsius z definiri.

144. Hoc autem modo plerumque difficulter Factores imaginarii eruuntur, quamobrem hoc Capite methodum peculiarem tradam, cujus ope sæpenumero Factores simplices imaginarii inveniri queant. Quoniam vero Factores simplices imaginarii ita sunt comparati, ut binorum productum fiat reale, hos ipsos Factores imaginarios reperiemus, si Factores investigemus duplices, seu hujus formæ $p - qz + rzz$, reales quidem, sed quorum Factores simplices sint imaginarii. Quod si enim Functionis $\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c.$, constent omnes Factores reales duplices hujus formæ trinomialis $p - qz + rzz$, simul omnes Factores imaginarii habebuntur.

145. Trinomium autem $p - qz + rzz$ Factores simplices habebit imaginarios, si fuerit $4pr > qq$ seu $\frac{q}{2\sqrt{pr}} < 1$. Cum igitur Sinus & Cosinus Angulorum sint unitate minores, formula $p - qz + rzz$ Factores simplices habebit imaginarios si fuerit $\frac{q}{2\sqrt{pr}}$ = Sinui vel Cosinui cujuspiam Anguli. Sit ergo $\frac{q}{2\sqrt{pr}} = cos. A\,\phi$, seu $q = 2\sqrt{pr}.\, cos.\,\phi$, atque trinomium $p - qz + rzz$ continebit Factores simplices imaginarios. Ne autem irrationalitas molestiam facessat, assumo hanc formam $pp - 2pqz.\,cos.\,\phi + qqzz$, cujus Factores simplices imaginarii erunt hi, $qz - p(cos.\,\phi + \sqrt{-1}.\, sin.\,\phi)$ & $qz - p(cos.\,\phi - \sqrt{-1}.\, sin.\,\phi)$. Ubi quidem patet si fuerit $cos.\,\phi = \pm 1$, tum ambos Factores, ob $sin.\,\phi = 0$, fieri æquales & reales.

146. Proposita ergo Functione integra $\alpha + \beta z + \gamma z^2 + \delta z^3 +$ &c.,

&c. , ejus Factores ſimplices imaginarii eruentur, ſi determinentur litteræ p & q cum Angulo ϕ, ut hoc trinomium $pp - 2pqz.\,coſ.\,\phi + qqzz$ fiat Factor Functionis. Tum enim ſimul inerunt iſti Factores ſimplices imaginarii $qz - p(coſ.\,\phi + \surd -1.\,ſin.\,\phi)$ & $qz - p(coſ.\,\phi - \surd -1.\,ſin.\,\phi)$. Quam ob rem Functio propoſita evaneſcet, ſi ponatur tam $z = \frac{p}{q} \times (coſ.\,\phi + \surd -1.\,ſin.\,\phi)$ quam $z = \frac{p}{q}(coſ.\,\phi - \surd -1.\,ſin.\,\phi)$. Hinc, facta ſubſtitutione utraque, duplex naſcetur æquatio, ex quibus tam fractio $\frac{p}{q}$ quam Arcus ϕ definiri poterunt. CAP. IX.

147. Hæ autem ſubſtitutiones loco z faciendæ, etiamſi primo intuitu difficiles videantur, tamen per ea, quæ in Capite præcedente ſunt tradita, ſatis expedite abſolventur. Cum enim fuerit oſtenſum eſſe $(coſ.\,\phi \pm \surd -1.\,ſin.\,\phi)^n = coſ.\,n\phi \pm \surd -1 \times ſin.\,n\phi$, ſequentes formulæ loco ſingularum ipſius z Poteſtatum habebuntur ſubſtituendæ.

pro priori Factore	pro altero Factore
$z = \frac{p}{q}(coſ.\,\phi + \surd -1.\,ſin.\,\phi)$	$z = \frac{p}{q}(coſ.\,\phi - \surd -1.\,ſin.\,\phi)$
$z^2 = \frac{p^2}{q^2}(coſ.\,2\phi + \surd -1.\,ſin.\,2\phi)$	$z^2 = \frac{p^2}{q^2}(coſ.\,2\phi - \surd -1.\,ſin.\,2\phi)$
$z^3 = \frac{p^3}{q^3}(coſ.\,3\phi + \surd -1.\,ſin.\,3\phi)$	$z^3 = \frac{p^3}{q^3}(coſ.\,3\phi - \surd -1.\,ſin.\,3\phi)$
$z^4 = \frac{p^4}{q^4}(coſ.\,4\phi + \surd -1.\,ſin.\,4\phi)$	$z^4 = \frac{p^4}{q^4}(coſ.\,4\phi - \surd -1.\,ſin.\,4\phi)$
&c.	&c.

Ponatur brevitatis gratia $\frac{p}{q} = r$, factaque ſubſtitutione ſequentes duæ naſcentur æquationes.

$$0 = \left\{ \begin{array}{l} \alpha + \beta r.\,coſ.\,\phi + \gamma r^2.\,coſ.\,2\phi + \delta r^3.\,coſ.\,3\phi + \&c. \\ + \beta r\surd -1.\,ſin.\,\phi + \gamma r^2\surd -1.\,ſin.\,2\phi + \delta r^3\surd -1.\,ſin.\,3\phi + \&c. \end{array} \right\}$$

$$0 = \left\{ \begin{array}{l} \alpha + \beta r.\,coſ.\,\phi + \gamma r^2.\,coſ.\,2\phi + \delta r^3.\,coſ.\,3\phi + \&c. \\ - \beta r\surd -1.\,ſin.\,\phi - \gamma r^2\surd -1.\,ſin.\,2\phi - \delta r^3\surd -1.\,ſin.\,3\phi - \&c. \end{array} \right\}$$

 148. Quod

148. Quod ſi hæ duæ æquationes invicem addantur & ſubtrahantur, & poſteriori caſu per $2\sqrt{-1}$ dividantur, prodibunt hæ duæ æquationes reales:

$$0 = \alpha + \beta r.cof.\Phi + \gamma r^2.cof.2\Phi + \delta r^3.cof.3\Phi + \&c.$$
$$0 = \beta r.fin.\Phi + \gamma r^2.fin.2\Phi + \delta r^3.fin.3\Phi + \&c.$$

quæ ſtatim ex forma Functionis propoſitæ

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \&c.$$

formari poſſunt, ponendo primum pro unaquaque ipſius z poteſtate $z^n = r^n\ cof.\,n\Phi$, deinceps $z^n = r^n\ fin.\,n\Phi$. Sic enim ob $fin.\,0\Phi = 0$ & $cof.\,0\Phi = 1$., pro z^0 ſeu 1 in termino conſtanti priori caſu ponitur 1, poſteriori autem 0. Si ergo ex his duabus æquationibus definiantur incognitæ r & Φ, ob $r = \frac{p}{q}$, habebitur Factor Functionis trinomialis $pp - 2pqz.\,cof.\Phi + qqzz$, duos Factores ſimplices imaginarios involvens.

149. Si æquatio prior multiplicetur per $fin.\,m\Phi$; poſterior per $cof.\,m\Phi$, atque producta vel addantur vel ſubtrahantur, prodibunt iſtæ duæ æquationes:

$$0 = \alpha.fin.m\Phi + \beta r.fin.(m+1)\Phi + \gamma r^2.fin.(m+2)\Phi + \delta r^3.fin.(m+3)\Phi + \&c.$$
$$0 = \alpha.fin.m\Phi + \beta r.fin.(m+1)\Phi + \gamma r^2.fin.(m-2)\Phi + \delta r^3.fin.(m-3)\Phi + \&c.$$

Sin autem æquatio prior multiplicetur per $cof.\,m\Phi$ & poſterior per $fin.\,m\Phi$, per additionem ac ſubtractionem ſequentes emergent æquationes.

$$0 = \alpha.cof.m\Phi + \beta r.cof.(m-1)\Phi + \gamma r^2.cof.(m-2)\Phi + \delta r^3.cof.(m-3)\Phi + \&c.$$
$$0 = \alpha.cof.m\Phi + \beta r.cof.(m+1)\Phi + \gamma r^2.cof.(m+2)\Phi + \delta r^3.cof.(m+3)\Phi + \&c.$$

Hujuſ-

Hujusmodi ergo duæ æquationes quæcunque conjunctæ determinabunt incognitas r & Φ; quod cum plerumque pluribus modis fieri possit, simul plures Factores trinomiales obtinentur, iique adeo omnes, quos Functio proposita in se complectitur.

150. Quo usus harum regularum clarius appareat, quarumdam Functionum sæpius occurrentium Factores trinomiales hic indagabimus, ut eos, quoties occasio postulaverit, hinc depromere liceat. Sit itaque proposita hæc Functio $a^n + z^n$, cujus Factores trinomiales formæ $pp - 2pqz.cos.\Phi + qq$ determinari oporteat; posito ergo $r = \frac{p}{q}$, habebuntur hæ duæ æquationes:

$0 = a^n + r^n.cos.n\Phi$ & $0 = r^n.sin.n\Phi$, quarum posterior dat $sin.n\Phi = 0$; unde erit $n\Phi$ Arcus vel hujus formæ $(2k+1)\pi$ vel $2k\pi$, denotante k numerum integrum. Casus hos ideo distinguo, quod eorum Cosinus sint differentes; priori enim casu erit $cos.(2k+1)\pi = -1$ posteriori casu autem $cos.2k\pi = +1$. Patet autem priorem formam $n\Phi = (2k+1)\pi$ sumi debere, quippe quæ dat $cos.n\Phi = -1$, unde fit $0 = a^n - r^n$, hincque porro $r = a = \frac{p}{q}$. Erit ergo $p = a$, $q = 1$, & $\Phi = \frac{(2k+1)\pi}{n}$, unde Functionis $a^n + z^n$ Factor erit $aa - 2az.cos.\frac{(2k+1)\pi}{n} + zz$. Cum igitur pro k numerum quemque integrum ponere liceat, prodeunt hoc modo plures Factores, neque tamen infiniti, quoniam si $2k+1$, ultra n augetur, Factores priores recurrunt, quod ex exemplis clarius patebit, cum sit $cos.(2\pi \pm \Phi) = cos.\Phi$. Deinde si n est numerus impar, posito $2k+1 = n$, erit Factor quadratus $aa + 2az + zz$ neque vero hinc sequitur quadratum $(a+z)^2$ esse Factorem Functionis $a^n + z^n$, quoniam (in §. 148) unica æquatio resultat, qua tantum patet $a+z$ esse Divisorem

formulæ

LIB. I. formulæ $a^n + z^n$; quæ regula ſemper eſt tenenda quoties $cof. \Phi$ ſit vel $+1$ vel -1.

EXEMPLUM.

Evolvamus aliquot caſus, quo iſti Factores clarius ob oculos ponantur, atque hos caſus in duas claſſes diſtribuamus, prout n fuerit numerus vel par vel impar.

Si $n = 1$ Formulæ $a + z$ Factor eſt $a + z$	Si $n = 2$ Formulæ $a^2 + z^2$ Factor eſt $a^2 + z^2$
Si $n = 3$ Formulæ $a^3 + z^3$ Factores ſunt $aa - 2az.cof.\frac{1}{3}\pi + zz$ $a + z$	Si $n = 4$ Formulæ $a^4 + z^4$ Factores ſunt $aa - 2az.cof.\frac{1}{4}\pi + zz$ $aa - 2az.cof.\frac{3}{4}\pi + zz$
Si $n = 5$ Formulæ $a^5 + z^5$ Factores ſunt $aa - 2az.cof.\frac{1}{5}\pi + zz$ $aa - 2az.cof.\frac{3}{5}\pi + zz$ $a + z$	Si $n = 6$ Formulæ $a^6 + z^6$ Factores ſunt $aa - 2az.cof.\frac{1}{6}\pi + zz$ $aa - 2az.cof.\frac{3}{6}\pi + zz$ $aa - 2az.cof.\frac{5}{6}\pi + zz$

Ex quibus exemplis patet omnes Factores obtineri, ſi loco $2k+1$ omnes numeri impares non majores, quam Exponens

n

n, substituantur, iis vero casibus quibus Factor quadratus prodit, tantum ejus radicem Factoribus annumerari debere. Cap. IX.

151. Si proposita sit hæc Functio $a^n - z^n$, ejus Factor trinomialis erit $pp - 2pqz.\,cos.\,\phi + qqzz$, si posito $r = \frac{p}{q}$, fuerit $0 = a^n - r^n.\,cos.\,n\phi$ & $0 = r^n.\,sin.\,n\phi$. Erit ergo iterum $sin.\,n\phi = 0$, ideoque $n\phi = (2k+1)\pi$ vel $n\phi = 2k\pi$. Hoc autem casu valor posterior sumi debet, ut sit $cos.\,n\phi = +1$, qui dat $0 = a^n - r^n$ & $r = \frac{p}{q} = a$. Habebitur itaque $p = a$; $q = 1$; & $\phi = \frac{2k\pi}{n}$; unde Factor trinomialis formulæ propositæ erit $= aa - 2az.\,cos.\,\frac{2k}{n}\pi + zz$; quæ forma, si loco $2k$ omnes numeri pares non majores quam n ponantur, simul dabit omnes Factores; ubi de Factoribus quadratis idem est tenendum quod ante monuimus. Ac primo quidem, posito $k = 0$, prodit Factor $aa - 2az + zz$, pro quo vero radix $a - z$ capi debet. Similiter, si n fuerit numerus par & ponatur $2k = n$, prodit $aa + 2az + zz$, unde patet $a + z$ esse divisorem formæ $a^n - z^n$

E X E M P L U M.

Casus Exponentis n ut ante tractati ita se habebunt, prout n fuerit numerus vel impar vel par.

Si $n = 1$ Formulæ $a - z$ ipſa erit Factor $a - z$	Si $n = 2$ Formulæ $a^2 - z^2$ Factores erunt $a - z$ $a + z$
Si $n = 3$ Formulæ $a^3 - z^3$ Factores erunt $a - z$ $aa - 2az.\cos.\frac{2}{3}\pi + zz$	Si $n = 4$ Formulæ $a^4 - z^4$ Factores erunt $a - z$ $aa - 2az.\cos.\frac{2}{4}\pi + zz$ $a + z$
Si $n = 5$ Formulæ $a^5 - z^5$ Factores erunt $a - z$ $aa - 2az.\cos.\frac{2}{5}\pi + zz$ $aa - 2az.\cos.\frac{4}{5}\pi + zz$	Si $n = 6$ Formulæ $a^6 - z^6$ Factores erunt $a - z$ $aa - 2az.\cos.\frac{2}{6}\pi + zz$ $aa - 2az.\cos.\frac{4}{6}\pi + zz$ $a + z$

152. His igitur confirmatur id, quod ſupra jam innuimus, omnem Functionem integram, ſi non in Factores ſimplices reales, tamen in Factores duplices reales reſolvi poſſe. Vidimus enim hanc Functionem indefinitæ dimenſionis $a^n + z^n$ ſemper in Factores duplices reales, præter ſimplices reales, reſolvi poſſe. Progrediamur ergo ad Functiones magis compoſitas, uti: $\alpha + \beta z^n + \gamma z^{2n}$, cujus quidem, ſi duos habeat Factores formæ $\eta + \theta z^n$, reſolutio ex præcedentibus abunde patet. Hoc ergo tantum erit efficiendum, ut formæ $\alpha + \beta z^n + \gamma z^{2n}$, eo caſu, quo non habet duos Factores reales formæ $\eta + \theta z^n$, reſo-

resolutionem in Factores reales, vel simplices vel duplices, doceamus.

153. Consideremus ergo hanc Functionem: $a^{2n} - 2a^n z^n \times cos. g + z^{2n}$, quæ in duos Factores formæ $\eta + \theta. z^n$ reales resolvi nequit. Quod si ergo ponamus hujus Functionis Factorem duplicem realem esse $pp - 2pqz. cos. \phi + qqzz$, posito $r = \frac{p}{q}$, duæ sequentes æquationes erunt resolvendæ: $0 = a^{2n} - 2a^n r^n \cdot cos. g. cos. n\phi + r^{2n}. cos. 2n\phi$ & $0 = - 2a^n r^n \cdot cos. g. sin. n\phi + r^{2n}. sin. 2n\phi$. Vel, loco prioris æquationis sumatur ex §. 149, (ponendo $m = 2n$), hæc $0 = a^{2n}. sin. 2n\phi - 2a^n r^n \cdot cos. g. sin. n\phi$, quæ cum posteriori collata dat $r = a$; tum vero erit $sin. 2n\phi = 2 cos. g. sin. n\phi$: At est $sin. 2n\phi = 2 sin. n\phi. cos. n\phi$. unde fit $cos. n\phi = cos. g$. At est semper $cos. (2k\pi \pm g) = cos. g$, ex quo habetur $n\phi = 2k\pi \pm g$ & $\phi = \frac{2k\pi \pm g}{n}$. Hinc ergo Factor generalis duplex formæ propositæ erit $= aa - 2az. cos. \frac{2k\pi \pm g}{n} + zz$; atque omnes Factores prodibunt, si pro $2k$ omnes numeri pares non majores quam n successive substituantur, uti ex applicatione ad casus videre licebit.

EXEMPLUM.

Consideremus ergo casus quibus n est 1, 2, 3, 4, &c., ut ratio Factorum appareat. Erit ergo

Formulæ
$a^2 - 2az. cos. g + zz$
Unicus Factor
$aa - 2az. cos. g + zz$

Formulæ
$a^4 - 2a^2z^2. cos. g + z^4$

 Facto-

LIB. I.

Factores duo

$$aa - 2az.\cos.\frac{g}{2} + z^2$$

$$aa - 2az\cos.(\frac{2\pi \pm g}{2} + zz \text{ seu } aa + 2az\cos.\frac{g}{2} + zz)$$

Formulæ

$$a^6 - 2a^3z^3.\cos.g + z^6$$

Factores tres

$$aa - 2az.\cos.\frac{g}{3} + z^2$$

$$aa - 2az.\cos.\frac{2\pi - g}{3} + z^2$$

$$aa - 2az.\cos.\frac{2\pi + g}{3} + z^2$$

Formulæ

$$a^8 - 2a^4.z^4\cos.g + z^8$$

Factores quatuor

$$aa - 2az.\cos.\frac{g}{4} + zz$$

$$aa - 2az.\cos.\frac{2\pi - g}{4} + zz$$

$$aa - 2az.\cos.\frac{2\pi + g}{4} + zz$$

$$aa - 2az.\cos.\frac{4\pi \pm g}{4} + zz \text{ seu } aa + 2az.\cos.\frac{g}{4} + zz$$

Formulæ

$$a^{10} - 2a^5z^5.\cos.g + z^{10}$$

Factores quinque

$$aa - 2az.\cos.\frac{g}{5} + zz$$

$$aa - 2az.\cos.\frac{2\pi - g}{5} + zz$$

$$aa - 2az.\cos.\frac{2\pi + g}{5} + zz$$

$$aa - 2az.\cos.\frac{4\pi - g}{5} + zz$$

$$aa - 2az.\cos.\frac{4\pi + g}{5} + zz$$

Con-

Confirmatur ergo etiam his exemplis omnem Functionem integram in Factores reales, sive simplices sive duplices, resolvi posse. Cap. IX.

154. Hinc ulterius progredi licebit ad Functionem hanc $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n}$, quæ certo habebit unum Factorem realem formæ $\eta + \theta z^n$, cujus igitur Factores reales, vel simplices vel duplices, exhiberi possunt; alter vero multiplicator formæ $\iota + \kappa z^n + \lambda z^{2n}$, utcumque fuerit comparatus, per §. præced. pari modo in Factores resolvi poterit. Deinde hæc Functio $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n}$, cum perpetuo habeat duos Factores reales formæ hujus $\eta + \theta z^n + \iota z^{2n}$, similiter in Factores, vel simplices vel duplices, reales resolvitur. Quin etiam progredi licet ad formam $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n} + \xi z^{5n}$ quæ, cum certo habeat unum Factorem formæ $\eta + \theta z^n$, alter Factor erit formæ præcedentis; unde etiam hæc Functio resolutionem in Factores reales, vel simplices vel duplices, admittet. Quare si ullum dubium mansisset circa hujusmodi resolutionem omnium Functionum integrarum, hoc nunc fere penitus tolletur.

155. Traduci vero etiam potest hæc in Factores resolutio ad Series infinitas; scilicet, quia vidimus supra esse $1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. = e^x$; at vero esse $e^x = (1 + \frac{x}{i})^i$, denotante i numerum infinitum, perspicuum est Seriem $1 + \frac{1}{x} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$ habere Factores infinitos simplices inter se æquales nempe $1 + \frac{x}{i}$. At si ab eadem Serie primus terminus dematur, erit $\frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} +$

Lib. I. &c. $= e^x - 1 = (1 + \frac{x}{i})^i - 1$, cujus formæ cum §. 151 comparatæ, quo fit $a = 1 + \frac{x}{i}$; $n = i$ & $z = 1$, Factor quicunque erit $= (1 + \frac{x}{i})^2 - 2(1 + \frac{x}{i}) cos. \frac{2k}{i}\pi + 1$, unde, substituendo pro $2k$ omnes numeros pares, simul omnes Factores prodibunt. Posito autem $2k = 0$ prodit Factor quadratus $\frac{xx}{ii}$, pro quo autem tantum ob rationes allegatas radix $\frac{x}{i}$ sumi debet, erit ergo x Factor expressionis $e^x - 1$ quod quidem sponte patet. Ad reliquos Factores inveniendos notari oportet esse, ob Arcum $\frac{2k}{i}\pi$ infinite parvum, $cos. \frac{2k}{i}\pi = 1 - \frac{2kk}{ii}\pi\pi$ (134), terminis sequentibus, ob i numerum infinitum, in nihilum abeuntibus. Hinc erit Factor quilibet $= \frac{xx}{ii} + \frac{4kk}{ii}\pi\pi + \frac{4kk\pi\pi}{i^3}x$, atque adeo forma $e^x - 1$ erit divisibilis per $1 + \frac{x}{i} + \frac{xx}{4kk\pi\pi}$. Quare expressio $e^x - 1 = x(1 + \frac{x}{1.2} + \frac{x^2}{1.2.3} + \frac{x^3}{1.2.3.4} + \&c.)$, præter Factorem x, habebit hos infinitos $(1 + \frac{x}{i} + \frac{xx}{4\pi\pi})(1 + \frac{x}{i} + \frac{xx}{16\pi\pi})(1 + \frac{x}{i} + \frac{xx}{36\pi\pi})(1 + \frac{x}{i} + \frac{xx}{64\pi\pi})$ &c.

156. Cum autem hi Factores contineant partem infinite parvam $\frac{x}{i}$, quæ, cum in singulis insit, atque per multiplicationem omnium, quorum numerus est $\frac{1}{2}i$, producat terminum $\frac{x}{2}$, omitti non potest. Ad hoc ergo incommodum vitandum consideremus hanc expressionem $e^x - e^{-x} =$

$$(1 + \frac{x}{i})^i - (1 - \frac{x}{i})^i = 2(\frac{x}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \&c.)$$

est

est enim $e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \&c.$; quæ cum Cap. IX.
§. 151. comparata dat $n = i$, $a = 1 + \frac{x}{i}$ & $z = 1 - \frac{x}{i}$ unde hujus expressionis Factor erit $= aa - 2az.cos.\frac{2k}{n}\pi + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii})\, cos.\frac{2k}{i}\pi = \frac{4xx}{ii} + \frac{4kk}{ii}\pi\pi - \frac{4kk\pi\pi xx}{i^4}$, ob $cos.\frac{2k}{i}\pi = 1 - \frac{2kk\pi\pi}{ii}$. Functio ergo $e^x - e^{-x}$ divisibilis erit per $1 + \frac{xx}{kk\pi\pi} - \frac{xx}{ii}$, ubi autem terminus $\frac{xx}{ii}$ tuto omittitur, quia etsi per i multiplicetur, tamen manet infinite parvus. Præterea vero ut ante, si $k = 0$, erit primus Factor $= x$. Quocirca, his Factoribus in ordinem redactis, erit $\frac{e^x - e^{-x}}{2} = x(1 + \frac{xx}{\pi\pi})(1 + \frac{xx}{4\pi\pi})(1 + \frac{xx}{9\pi\pi})(1 + \frac{xx}{16\pi\pi})(1 + \frac{xx}{25\pi\pi})\,\&c. = x(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.\,..7} + \&c.)$. Singulis scilicet Factoribus per multiplicationem constantis ejusmodi formam dedi, ut per actualem multiplicationem primus terminus x resultet.

157. Eodem modo cum sit $\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \&c. = \frac{(1 + \frac{x}{i})^i + (1 - \frac{x}{i})^i}{2}$, hujus expressionis cum superiori $a^n + z^n$ comparatio dabit $a = 1 + \frac{x}{i}$; $z = 1 - \frac{x}{i}$ & $n = i$: erit ergo Factor quicunque $= aa - 2az \times cos.\frac{2k+1}{n}\pi + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii})\, cos.\frac{2k+1}{i}\pi$. Est autem

LIB. I. autem $cos. \frac{2k+1}{i}\pi = 1 - \frac{(2k+1)^2\pi\pi}{2ii}$, unde forma Factoris erit $= \frac{4xx}{ii} + \frac{(2k+1)^2\pi^2}{ii}$, evanescente termino cujus denominator est i^4. Quoniam ergo omnis Factor expressionis $1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} +$ &c. hujusmodi formam habere debet $1 + \alpha xx$, quo Factor inventus ad hanc formam reducatur, dividi debet per $\frac{(2k+1)^2\pi^2}{ii}$: hinc Factor formæ propositæ erit $1 + \frac{4xx}{(2k+1)^2\pi\pi}$, ex eoque omnes Factores infiniti invenientur, si loco $2k+1$ successive omnes numeri impares substituantur. Hanc ob rem erit

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \&c. =$$

$$\left(1 + \frac{4xx}{\pi\pi}\right)\left(1 + \frac{4xx}{9\pi\pi}\right)\left(1 + \frac{4xx}{25\pi\pi}\right)\left(1 + \frac{4xx}{49\pi\pi}\right) \&c.$$

158. Si x fiat quantitas imaginaria, formulæ hæ exponentiales in Sinum & Cosinum cujuspiam Arcus realis abeunt. Sit enim $x = z\sqrt{-1}$; erit $\frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}} =$ $fin.\ z = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3 \ldots 7} + \&c.$, quæ adeo expressio hos habet Factores numero infinitos $z\left(1 - \frac{zz}{\pi\pi}\right)\left(1 - \frac{zz}{4\pi\pi}\right)\left(1 - \frac{zz}{9\pi\pi}\right)\left(1 - \frac{zz}{16\pi\pi}\right)\left(1 - \frac{zz}{25\pi\pi}\right)$ &c., seu erit $fin.\ z = z\left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)$ $\left(1 + \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{3\pi}\right)$ &c.. Quoties ergo Arcus z ita est comparatus, ut quispiam Factor evanescat, quod fit si $z = 0$, $z = \pm\pi$; $z = \pm 2\pi$, & generaliter si $z = \pm k\pi$, denotante k numerum quemcunque integrum, simul Sinus

nus ejus Arcus debet esse $= 0$, quod quidem ita patet, ut hinc istos Factores a posteriori eruere licuisset. Cap. IX.

Simili modo, cum sit $\frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2} = cos.\, z$, erit quoque $cos.\, z = (1 - \frac{4zz}{\pi\pi})(1 - \frac{4zz}{9\pi\pi})(1 - \frac{4zz}{25\pi\pi})(1 - \frac{4zz}{49\pi\pi})$ &c., seu, his Factoribus in binos resolvendis, erit quoque $cos.\, z = (1 - \frac{2z}{\pi})(1 + \frac{2z}{\pi})(1 - \frac{2z}{3\pi})(1 + \frac{2z}{3\pi})(1 - \frac{2z}{5\pi})(1 + \frac{2z}{5\pi})$ &c., ex qua pari modo patet, si fuerit $z = \pm \frac{(2k+1)}{2}\pi$, fore $cos.\, z = 0$, id quod etiam ex natura Circuli liquet.

159. Ex §. 152. etiam inveniri possunt Factores hujus expressionis $e^{x} - 2\, cos.\, g + e^{-x} = 2(1 - cos.\, g + \frac{xx}{1.\,2} + \frac{x^4}{1.\,2.\,3.\,4} +$ &c. $)$. Transit enim hæc expressio in hanc $(1 + \frac{x}{i})^{i} - 2\, cos.\, g + (1 - \frac{x}{i})^{i}$, quæ cum illa forma comparata dat $2n = i$; $a = 1 + \frac{x}{i}$, & $z = 1 - \frac{x}{i}$, unde Factor quicunque hujus formulæ erit $= aa - 2az.\, cos.\, \frac{2k\pi \pm g}{n} + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii}).\, cos.\, \frac{2(2k\pi \pm g)}{i}$: at est $cos.\, \frac{2(2k\pi \pm g)}{i} = 1 - \frac{2(2k\pi \pm g)^2}{ii}$, unde Factor erit $= \frac{4xx}{ii} + \frac{4(2k\pi \pm g)^2}{ii}$, seu hujus formæ $1 + \frac{xx}{(2k\pi \pm g)^2}$. Si ergo expressio per $2(1 - cos.\, g)$ dividatur, ut in Serie infinita terminus constans sit $= 1$, erit, sumendis omnibus Factoribus, $\frac{e^{x} - 2\, cos.\, g + e^{-x}}{2(1 - cos.\, g)} = (1 + \frac{xx}{gg})(1 + \frac{xx}{(2\pi - g)^2})$

 (1 +

Lib. I. $(1+\frac{xx}{(2\pi+g)^2})(1+\frac{xx}{(4\pi-g)^2})(1+\frac{xx}{(4\pi+g)^2})$ $(1+\frac{xx}{(6\pi-g)^2})(1+\frac{xx}{(6\pi+g)^2})$ &c.. Atque, si loco x ponatur $z\sqrt{-1}$, erit $\frac{cos.z-cos.g}{1-cos.g}=(1-\frac{z}{g})(1+\frac{z}{g})$ $(1-\frac{z}{2\pi-g})(1+\frac{z}{2\pi-g})(1-\frac{z}{2\pi+g})(1+\frac{z}{2\pi+g})$ $(1-\frac{z}{4\pi-g})(1+\frac{z}{4\pi-g})$ &c., $=1-\frac{zz}{1.2(1-cos.g)}+\frac{z^4}{1.2.3.4(1-cos.g)}-\frac{z^6}{1.2\ldots 6(1-cos.g)}+$ &c.. Hujus adeo Seriei in infinitum continuatæ Factores omnes cognoscuntur.

160. Commode etiam hujusmodi Functionis $e^{b+x}\pm e^{c-x}$ Factores inveniri omnesque assignari possunt. Transmutatur enim in hanc formam $(1+\frac{b+x}{i})^i\pm(1+\frac{c-x}{i})^i$, quæ comparata cum forma $a^i\pm z^i$, Factorem habebit $aa-2az.cos.\frac{m\pi}{i}+zz$, denotante m numerum imparem si valeat signum superius, contra vero numerum parem. Cum autem, ob i numerum infinite magnum, sit $cos.\frac{m\pi}{i}=1-\frac{mm\pi\pi}{2ii}$, erit Factor ille generalis $=(a-z)^2+\frac{mm\pi\pi}{ii}a.z.$ At hoc casu erit $a=1+\frac{b+x}{i}$ & $z=1+\frac{c-x}{i}$, unde fit $(a-z)^2=\frac{(b-c+2x)^2}{ii}$ & $az=1+\frac{b+c}{i}+\frac{bc+(c-b)x-xx}{ii}$; ideoque Factor erit per ii multiplicatus $=(b-c)^2+4(b-c)x+4xx+mm\pi\pi$, neglectis terminis per i vel ii divisis, quoniam jam omnis generis termini adsunt, præ quibus hi evanescerent. Termino ergo constante ad unitatem per divisionem reducto erit Factor $=1+\frac{4(b-c)x+4xx}{mm\pi\pi+(b-c)^2}$.

161. Nunc,

161. Nunc, quoniam in omnibus Factoribus terminus constans est $= 1$, ipsa Functio $e^{b+x} \pm e^{c-x}$ per ejusmodi constantem dividi debet, ut terminus constans fiat $= 1$, seu ut ejus valor, posito $x = 0$, fiat $= 1$; talis Divisor erit $e^b \pm e^c$, & hanc ob rem expressio hæc $\frac{e^{b+x} \pm e^{c-x}}{e^b \pm e^c}$ per Factores numero infinitos exponi poterit. Erit ergo, si valeat signum superius atque m denotet numerum imparem, Cap. IX.

$$\frac{e^{b+x} + e^{c-x}}{e^b + e^c} = \left(1 + \frac{4(b-c)x + 4xx}{\pi\pi + (b-c)^2}\right)\left(1 + \frac{4(b-c)x + 4xx}{9\pi\pi + (b-c)^2}\right)$$

$\left(1 + \frac{4(b-c)x + 4xx}{25\pi\pi + (b-c)^2}\right)$ &c., sin autem signum inferius valeat, atque ideo m denotet numerum parem, casuque $m = 0$ radix Factoris quadrati ponatur, erit $\frac{e^{b+x} - e^{c-x}}{e^b - e^c} =$

$$\left(1 + \frac{2x}{b-c}\right)\left(1 + \frac{4(b-c)x + 4xx}{4\pi\pi + (b-c)^2}\right)\left(1 + \frac{4(b-c)x + 4xx}{16\pi\pi + (b-c)^2}\right)$$

$\left(1 + \frac{4(b-c)x + 4xx}{36\pi\pi + (b-c)^2}\right)$ &c.

162. Ponatur $b = 0$, quod sine detrimento universalitatis fieri potest, eritque $\frac{e^x + e^c . e^{-x}}{1 + e^c} = \left(1 - \frac{4cx + 4xx}{\pi\pi + cc}\right)$

$\left(1 - \frac{4cx + 4xx}{9\pi\pi + cc}\right)\left(1 - \frac{4cx + 4xx}{25\pi\pi + cc}\right)$ &c.; $\frac{e^x - e^c . e^{-x}}{1 - e^c}$

$= \left(1 - \frac{2x}{c}\right)\left(1 - \frac{4cx + 4xx}{4\pi\pi + cc}\right)\left(1 - \frac{4cx + 4xx}{16\pi\pi + cc}\right)$

$\left(1 - \frac{4cx + 4xx}{36\pi\pi + cc}\right)$ &c.. Jam ponatur c negativum, atque habebuntur hæ duæ æquationes: $\frac{e^x + e^{-c} . e^{-x}}{1 + e^{-c}} =$

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$(1 + \frac{4cx + 4xx}{\pi\pi + cc})(1 + \frac{4cx + 4xx}{9\pi\pi + cc})(1 + \frac{4cx + 4xx}{25\pi\pi + cc})$ &c.,

$$\frac{e^{x} - e^{-c} e^{-x}}{1 - e^{-c}} = (1 + \frac{2x}{c})(1 + \frac{4cx + 4xx}{4\pi\pi + cc})$$

$(1 + \frac{4cx + 4xx}{16\pi\pi + cc})(1 + \frac{4cx + 4xx}{36\pi\pi + cc})$ &c.. Multiplicetur forma prima per tertiam, ac prodibit $\frac{e^{2x} + e^{-2x} + e^{c} + e^{-c}}{2 + e^{c} + e^{-c}}$; ponatur vero y loco $2x$, eritque $\frac{e^{y} + e^{-y} + e^{c} + e^{-c}}{2 + e^{c} + e^{-c}} = (1 - \frac{2cy + yy}{\pi\pi + cc})$ $(1 + \frac{2cy + yy}{\pi\pi + cc})(1 - \frac{2cy + yy}{9\pi\pi + cc})(1 + \frac{2cy + yy}{9\pi\pi + cc})(1 - \frac{2cy + yy}{25\pi\pi + cc})$ $(1 + \frac{2cy + yy}{25\pi\pi + cc})$ &c.. Multiplicetur prima forma per quartam, erit productum $= \frac{e^{2x} - e^{-2x} + e^{c} - e^{-c}}{e^{c} - e^{-c}}$; ponatur y pro $2x$, eritque $\frac{e^{y} - e^{-y} + e^{c} - e^{-c}}{e^{c} - e^{-c}} =$ $(1 + \frac{y}{c})(1 - \frac{2cy + yy}{\pi\pi + cc})(1 + \frac{2cy + yy}{4\pi\pi + cc})(1 - \frac{2cy + yy}{9\pi\pi + cc})$ $(1 + \frac{2cy + yy}{16\pi\pi + cc})(1 - \frac{2cy + yy}{25\pi\pi + cc})$ &c.. Si secunda forma per quartam multiplicetur, prodibit eadem æquatio nisi quod c capiendum sit negativum, erit nempe

$$\frac{e^{c} - e^{-c} - e^{y} + e^{-y}}{e^{c} - e^{-c}} = (1 - \frac{y}{c})(1 + \frac{2cy + yy}{\pi\pi + cc})$$

$(1 - \frac{2cy + yy}{4\pi\pi + cc})(1 + \frac{2cy + yy}{9\pi\pi + cc})(1 - \frac{2cy + yy}{16\pi\pi + cc})$ $(1 + \frac{2cy + yy}{25\pi\pi + cc})(1 - \frac{2cy + yy}{36\pi\pi + cc})$ &c.. Multiplicetur denique

nique forma secunda per quartam eritque $\frac{e^y + e^{-y} - e^c - e^{-c}}{2 - e^c - e^{-c}}$ $= (1 - \frac{yy}{cc})(1 - \frac{2cy + yy}{4\pi\pi + cc})(1 + \frac{2cy + yy}{4\pi\pi + cc})(1 - \frac{2cy + yy}{16\pi\pi + cc})$ $(1 + \frac{2cy + yy}{16\pi\pi + cc})(1 - \frac{2cy + yy}{36\pi\pi + cc})(1 + \frac{2cy + yy}{36\pi\pi + cc})$ &c.

163. Hæ quatuor combinationes nunc commode ad Circulum transferri possunt, ponendo $c = g\sqrt{-1}$ & $y = v\sqrt{-1}$: erit enim $e^{v\sqrt{-1}} + e^{-v\sqrt{-1}} = 2\, cos.\, v$; $e^{v\sqrt{-1}} - e^{-v\sqrt{-1}} = 2\sqrt{-1}.\, fin.\, v.$ & $e^{g\sqrt{-1}} + e^{-g\sqrt{-1}} = 2\, cos.\, g$; $e^{g\sqrt{-1}} - e^{-g\sqrt{-1}} = 2\sqrt{-1}.\, fin.\, g.$ Hinc prima combinatio dabit $\frac{cos.\, v + cos.\, g}{1 + cos.\, g} = 1 - \frac{vv}{1.2(1 + cos.\, g)} +$ $\frac{v^4}{1.2.3.4(1 + cos.\, g)} - \frac{v^6}{1.2....6(1 + cos.\, g)} +$ &c. $= (1 + \frac{2gv - vv}{\pi\pi - gg})$ $(1 - \frac{2gv - vv}{\pi\pi - gg})(1 + \frac{2gv - vv}{9\pi\pi - gg})(1 - \frac{2gv - vv}{9\pi\pi - gg})$ $(1 + \frac{2gv - vv}{25\pi\pi - gg})(1 - \frac{2gv - vv}{25\varpi\varpi - gg})$ &c. $= (1 + \frac{v}{\varpi - g})$ $(1 - \frac{v}{\varpi + g})(1 - \frac{v}{\varpi - g})(1 + \frac{v}{\varpi + g})(1 + \frac{v}{3\varpi - g})$ $(1 - \frac{v}{3\varpi + g})(1 - \frac{v}{3\varpi - g})(1 + \frac{v}{3\varpi + g})$ &c. $=$ $(1 - \frac{vv}{(\varpi - g)^2})(1 - \frac{vv}{(\varpi + g)^2})(1 - \frac{vv}{(3\varpi - g)^2})$ $(1 - \frac{vv}{(3\varpi + g)^2})(1 - \frac{vv}{(5\varpi - g)^2})$ &c.. Quarta vero combinatio dat $\frac{cos.\, v - cos.\, g}{1 - cos.\, g} = 1 - \frac{vv}{1.2(1 - cos.\, g)} +$ $\frac{v^4}{1.2.3.4(1 - cos.\, g)} - \frac{v^6}{1.2.....6(1 - cos.\, g)} +$ &c. $= (1 - \frac{vv}{gg})$ $(1 + \frac{2gv - vv}{4\varpi\varpi - gg})(1 - \frac{2gv - vv}{4\varpi\varpi - gg})(1 + \frac{2gv - vv}{16\varpi\varpi - gg})$

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$$\left(1-\frac{2gv-vv}{16\varpi\varpi-gg}\right) \&c. = \left(1-\frac{v}{g}\right)\left(1+\frac{v}{g}\right)\left(1+\frac{v}{2\varpi-g}\right)$$
$$\left(1-\frac{v}{2\varpi+g}\right)\left(1-\frac{v}{2\varpi-g}\right)\left(1+\frac{v}{2\varpi+g}\right)\left(1+\frac{v}{4\varpi-g}\right)$$
$$\left(1-\frac{v}{4\varpi+g}\right) \&c. = \left(1-\frac{vv}{gg}\right)\left(1-\frac{vv}{(2\varpi-g)^2}\right)$$
$$\left(1-\frac{vv}{(2\varpi+g)^2}\right)\left(1-\frac{vv}{(4\varpi-g)^2}\right)\left(1-\frac{vv}{(4\varpi+g)^2}\right) \&c..$$

Secunda combinatio dat $\frac{\text{fin. } g+\text{fin. } v}{\text{fin. } g} = 1+\frac{v}{\text{fin.} g}-\frac{v^3}{1.2.3\,\text{fin.} g}+$
$$\frac{v^5}{1.2.\ldots5\,\text{fin. } g} - \&c. = \left(1+\frac{v}{g}\right)\left(1+\frac{2gv-vv}{\varpi\varpi-gg}\right)$$
$$\left(1-\frac{2gv-vv}{4\varpi\varpi-gg}\right)\left(1+\frac{2gv-vv}{9\varpi\varpi-gg}\right)\left(1-\frac{2gv-vv}{16\varpi\varpi-gg}\right) \&c.$$
$$=\left(1+\frac{v}{g}\right)\left(1+\frac{v}{\varpi-g}\right)\left(1-\frac{v}{\varpi+g}\right)\left(1-\frac{v}{2\varpi-g}\right)$$
$$\left(1+\frac{v}{2\varpi+g}\right)\left(1+\frac{v}{3\varpi-g}\right)\left(1-\frac{v}{3\varpi+g}\right)\left(1-\frac{v}{5\varpi-g}\right) \&c..$$
Ac fumto v negativo prodit tertia combinatio.

164. Ipfæ vero etiam expreffiones in §. 162. primum inventæ ad Arcus circulares traduci poffunt hoc modo: cum fit
$$\frac{e^x+e.^c e^{-x}}{1+e^c} = \frac{(1+e^{-c})(e^x+e.^c e^{-x})}{2+e^c+e^{-c}} =$$
$$\frac{e^x+e^{-x}+e^{c-x}+e^{-c+x}}{2+e^c+e^{-c}},$$ fi ponamus $c=g\sqrt{-1}$ & $x=z\sqrt{-1}$, hæc expreffio abit in hanc $\frac{\text{cof.} z+\text{cof.}(g-z)}{1+\text{cof.} g} =$ $\text{cof.} z+\frac{\text{fin. } g.\,\text{fin. } z}{1+\text{cof.} g}$. Erit ergo (ob $\frac{\text{fin. } g}{1+\text{cof.} g}=\text{tang.}\frac{1}{2}g$)
$$\text{cof.} z+\text{tang.}\frac{1}{2}g.\text{fin.} z = 1+\frac{z}{1}\text{tang.}\frac{1}{2}g-\frac{zz}{1.2}-$$
$$\frac{z^3}{1.2.3}\text{tang.}\frac{1}{2}g+\frac{z^4}{1.2.3.4}+\frac{z^5}{1.2.3.4.5}\text{tang.}\frac{1}{2}g - \&c.$$
$$=\left(1+\frac{4gz-4zz}{\varpi\varpi-gg}\right)\left(1+\frac{4gz-4zz}{9\varpi\varpi-gg}\right)\left(1+\frac{4gz-4zz}{25\varpi\varpi-gg}\right) \&c.$$
$$=(1+$$

$= (1 + \frac{2z}{\varpi - g})(1 - \frac{2z}{\varpi + g})(1 + \frac{2z}{3\varpi - g})(1 - \frac{2z}{3\varpi + g})$ $(1 + \frac{2z}{5\varpi - g})(1 - \frac{2z}{5\varpi + g})$ &c.. Simili modo altera expressio, si Numerator & Denominator per $1 - e^{-c}$ multiplicetur, abit in $\frac{e^x + e^{-x} - e^{c-x} - e^{-c+x}}{2 - e^c - e^{-c}}$; quæ, facto $c = g\sqrt{-1}$ & $x = z\sqrt{-1}$, dat $\frac{cos. z - cos. (g - z)}{1 - cos. g} =$ $cos. z - \frac{sin. g. sin. z.}{1 - cos. g} = cos. z - \frac{sin. z}{tang. \frac{1}{2} g}$. Erit ergo $cos. z -$ $cot. \frac{1}{2} g. sin. z = 1 - \frac{z}{1} cot. \frac{1}{2} g - \frac{zz}{1.2} + \frac{z^3}{1.2.3} cot. \frac{1}{2} g +$ $\frac{z^4}{1.2.3.4} - \frac{z^5}{1.2...5} cot. \frac{1}{2} g + \&c. = (1 - \frac{2z}{g})(1 + \frac{4gz - 4zz}{4\varpi\varpi - gg})$ $(1 + \frac{4gz - 4zz}{16\varpi\varpi - gg})(1 + \frac{4gz - 4zz}{36\pi\pi - gg})\ \&c. = (1 - \frac{2z}{g})$ $(1 + \frac{2z}{2\pi - g})(1 - \frac{2z}{2\pi + g})(1 + \frac{2z}{4\pi - g})(1 - \frac{2z}{4\pi + g})$ &c.

Quod si ergo ponatur $v = 2z$ seu $z = \frac{1}{2} v$; habebitur

$$\frac{cos. \frac{1}{2}(g - v)}{cos. \frac{1}{2} g} = cos. \frac{1}{2} v + tang. \frac{1}{2} g. sin. \frac{1}{2} v =$$
$$(1 + \frac{v}{\varpi - g})(1 - \frac{v}{\varpi + g})(1 + \frac{v}{3\varpi - g})(1 - \frac{v}{3\varpi + g})\&c.;$$
$$\frac{cos. \frac{1}{2}(g + v)}{cos. \frac{1}{2} g} = cos. \frac{1}{2} v - tang. \frac{1}{2} g. sin. \frac{1}{2} v =$$
$$(1 - \frac{v}{\varpi - g})(1 + \frac{v}{\varpi + g})(1 - \frac{v}{3\varpi - g})(1 + \frac{v}{3\varpi + g})\&c.;$$
$$\frac{sin. \frac{1}{2}(g - v)}{sin. \frac{1}{2} g} = cos. \frac{1}{2} v - cot. \frac{1}{2} g. sin. \frac{1}{2} v =$$
$$(1 - \frac{v}{g})(1 + \frac{v}{2\varpi - g})(1 - \frac{v}{2\varpi + g})(1 + \frac{v}{4\varpi - g})\ \&c.$$

$\frac{sin. \frac{1}{2}(g + v)}{sin. \frac{1}{2} g}$

LIB. I. $\frac{\sin.\frac{1}{2}(g+v)}{\sin.\frac{1}{2}g} = \cos.\frac{1}{2}v + \cot.\frac{1}{2}g.\sin.\frac{1}{2}v =$

$$(1+\frac{v}{g})(1-\frac{v}{2\varpi-g})(1+\frac{v}{2\varpi+g})(1-\frac{v}{4\varpi-g})\ \&c.$$

Quorum Factorum lex progreſſionis ſatis eſt ſimplex & uniformis; atque ex his expreſſionibus per multiplicationem oriuntur eæ ipſæ, quæ §. præcedente ſunt inventæ.

CAPUT X.

De uſu Factorum inventorum in definiendis ſummis Serierum infinitarum.

165. SI fuerit $1 + Az + Bz^2 + Cz^3 + Dz^4 + \&c. = (1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)\ \&c.$, hi Factores, ſive ſint numero finiti ſive infiniti, ſi in ſe actu multiplicentur, illam expreſſionem $1 + A + Bz^2 + Cz^3 + Dz^4 + \&c.$, producere debent. Æquabitur ergo coëfficiens A ſummæ omnium quantitatum $\alpha + \beta + \gamma + \delta + \varepsilon + \&c.$. Coëfficiens vero B æqualis erit ſummæ productorum ex binis, eritque $B = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta + \&c.$. Tum vero coëfficiens C æquabitur ſummæ productorum ex ternis, nempe erit $C = \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta + \&c.$. Atque ita porro erit $D =$ ſummæ productorum ex quaternis, $E =$ ſummæ productorum ex quinis, &c., id quod ex Algebra communi conſtat.

166. Quia ſumma quantitatum $\alpha + \beta + \gamma + \delta + \&c.$, datur una cum ſumma productorum ex binis, hinc ſumma Quadratorum $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c.$, inveniri poterit, quippe quæ æqualis eſt Quadrato ſummæ demtis duplicibus productis ex binis. Simili modo ſumma Cuborum, Biquadratorum & altiorum Poteſtatum definiri poteſt: ſi enim ponamus

$P =$

$$P = \alpha + \beta + \gamma + \delta + \varepsilon + \&c.$$
$$Q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \&c.$$
$$R = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3 + \&c.$$
$$S = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4 + \&c.$$
$$T = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5 + \&c.$$
$$V = \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \varepsilon^6 + \&c.$$
&c.

Valores P, Q, R, S, T, V &c. ſequenti modo ex cognitis A, B, C, D, &c., determinabuntur.

$$P = A$$
$$Q = AP - 2B$$
$$R = AQ - BP + 3C$$
$$S = AR - BQ + CP - 4D$$
$$T = AS - BR + CQ - DP + 5E$$
$$V = AT - BS + CR - DQ + EP - 6F$$
&c.

quarum formularum veritas examine inſtituto facile agnoſcitur: interim tamen in calculo differentiali ſummo cum rigore demonſtrabitur.

167. Cum igitur ſupra (§. 156.) invenerimus eſſe: $\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.\ldots.7} + \&c.\right) = x\left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{xx}{25\pi\pi}\right)\&c.$, erit $1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.3\ldots7} + \&c. = \left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\&c.$ Ponatur $xx = \pi\pi z$, eritque $1 + \frac{\pi\pi}{1.2.3}z + \frac{\pi^4}{1.2.3.4.5}z^2 + \frac{\pi^6}{1.2.3\ldots.7}z^3 + \&c. = (1+z)\left(1+\frac{1}{4}z\right)\left(1+\frac{1}{9}z\right)\left(1+\frac{1}{16}z\right)$

 (1 +

LIB. I. $(1+\frac{1}{25}z)$ &c.. Facta ergo applicatione superioris regulæ ad hunc casum, erit $A=\frac{\pi\pi}{6}$; $B=\frac{\pi^4}{120}$; $C=\frac{\pi^6}{5040}$; $D=\frac{\pi^8}{362880}$ &c.. Quod si ergo ponatur

$$P=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\text{\&c.}$$
$$Q=1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}+\frac{1}{36^2}+\text{\&c.}$$
$$R=1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\frac{1}{25^3}+\frac{1}{36^3}+\text{\&c.}$$
$$S=1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\frac{1}{25^4}+\frac{1}{36^4}+\text{\&c.}$$
$$T=1+\frac{1}{4^5}+\frac{1}{9^5}+\frac{1}{16^5}+\frac{1}{25^5}+\frac{1}{36^5}+\text{\&c.}$$
&c.

atque harum litterarum valores ex A, B, C, D, &c. determinentur, prodibit.

$$P=\frac{\pi\pi}{6}$$
$$Q=\frac{\pi^4}{90}$$
$$R=\frac{\pi^6}{945}$$
$$S=\frac{\pi^8}{9450}$$
$$T=\frac{\pi^{10}}{93555}$$
&c.

168. Patet ergo omnium Serierum infinitarum in hac forma generali $1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+$ &c., contentarum, quoties n fuerit numerus par, ope Peripheriæ Circuli π exhiberi posse; habebit enim semper summa Seriei ad π^n rationem rationalem.

lem. Quo autem valor harum ſummarum clarius perſpiciatur, plures hujuſmodi Serierum ſummas commodiori modo expreſſas hic adjiciam. Cap. X.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\&c.=\frac{2^0}{1.2.3}.\frac{1}{1}\pi^2$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\&c.=\frac{2^2}{1.2.3.4.5}.\frac{1}{3}\pi^4$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\&c.=\frac{2^4}{1.2.3....7}.\frac{1}{3}\pi^6$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\&c.=\frac{2^6}{1.2.3....9}.\frac{3}{5}\pi^8$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\&c.=\frac{2^8}{1.2.3.....11}.\frac{5}{3}\pi^{10}$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\&c.=\frac{2^{10}}{1.2.3.....13}.\frac{691}{105}\pi^{12}$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\&c.=\frac{2^{12}}{1.2.3.....15}.\frac{35}{1}\pi^{14}$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\&c.=\frac{2^{14}}{1.2.3.....17}.\frac{3617}{15}\pi^{16}$$

$$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\&c.=\frac{2^{16}}{1.2.3.....19}.\frac{43867}{21}\pi^{18}$$

$$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\&c.=\frac{2^{18}}{1.2.3....21}.\frac{1222277}{55}\pi^{20}$$

$$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\&c.=\frac{2^{20}}{1.2.3...23}.\frac{854513}{3}\pi^{22}$$

$$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\&c.=\frac{2^{22}}{1.\ 2.\ 3.........25}.\frac{1181820455}{273}\pi^{24}$$

$$1+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\frac{1}{4^{26}}+\frac{1}{5^{26}}+\&c.=\frac{2^{24}}{1.\ 2.\ 3.........27}.\frac{76977927}{1}\pi^{26}.$$

Hucuſque iſtos Poteſtatum ipſius π Exponentes artificio alibi exponendo continuare licuit, quod ideo hic adjunxi, quod

 Seriei

LIB. I. Seriei fractionum primo intuitu perquam irregularis 1, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{3}{5}$, $\frac{5}{3}$, $\frac{691}{105}$, $\frac{35}{1}$, &c. in plurimis occasionibus eximius est usus.

169. Tractemus eodem modo æquationem §. 157. inventam, ubi erat $\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} +$ &c., $= (1 + \frac{4xx}{\pi\pi})(1 + \frac{4xx}{9\pi\pi})(1 + \frac{4xx}{25\pi\pi})(1 + \frac{4xx}{49\pi\pi})$ &c.. Posito ergo $xx = \frac{\pi\pi z}{4}$ erit $1 + \frac{\pi\pi}{1.2.4}z + \frac{\pi^4}{1.2.3.4.4^2}zz + \frac{\pi^6}{1.2 \ldots 6.4^3}z^3 +$ &c., $= (1 + z)(1 + \frac{1}{9}z)(1 + \frac{1}{25}z)(1 + \frac{1}{49}z)$ &c.. Unde, facta applicatione, erit $A = \frac{\pi\pi}{1.2.4}$; $B = \frac{\pi^4}{1.2.3.4.4^2}$; $C = \frac{\pi^6}{1.2.3 \ldots 6.4^3}$; &c.. Quod si ergo ponamus

$$P = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \&c.$$

$$Q = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \frac{1}{81^2} + \&c.$$

$$R = 1 + \frac{1}{9^3} + \frac{1}{25^3} + \frac{1}{49^3} + \frac{1}{81^3} + \&c.$$

$$S = 1 + \frac{1}{9^4} + \frac{1}{25^4} + \frac{1}{49^4} + \frac{1}{81^4} + \&c.$$

&c.

reperientur sequentes pro P, Q, R, S, &c., valores:

$$P = \frac{1}{1} \cdot \frac{\pi^2}{2^3}; \qquad Q = \frac{2}{1.2.3} \cdot \frac{\pi^4}{2^5}$$

$$R = \frac{16}{1.2.3.4.5} \cdot \frac{\pi^6}{2^7}; \qquad S = \frac{272}{1.2.3 \ldots 7} \cdot \frac{\pi^8}{2^9}$$

$T =$

$$T = \frac{7936}{1.2.3\ldots\ldots 9} \cdot \frac{\pi^{10}}{2^{11}}; \; V = \frac{353792}{1.2.3\ldots\ldots 11} \cdot \frac{\pi^{12}}{2^{13}}$$ Cap. X.

$$W = \frac{22368256}{1.2.3\ldots\ldots\ldots 13} \cdot \frac{\pi^{14}}{2^{15}}.$$

170. Eædem summæ Potestatum numerorum imparium inveniri possunt ex summis præcedentibus, in quibus omnes numeri occurrunt; si enim fuerit $M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} +$ &c., erit ubique, per $\frac{1}{2^n}$ multiplicando, $\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} +$ &c., quæ Series numeros tantum pares continens, si a priori subtrahatur, relinquet numeros impares, eritque ideo $M - \frac{M}{2^n} = \frac{2^n - 1}{2^n} M = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} +$ &c.. Quod si autem Series $\frac{M}{2^n}$ bis sumta subtrahatur ab M signa prodibunt alternantia, eritque $M - \frac{2M}{2^n} = \frac{2^{n-1} - 1}{2^{n-1}} M = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} +$ &c.. Per tradita ergo præcepta summari poterunt hæ Series

$$1 \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \frac{1}{4^n} + \frac{1}{5^n} \pm \frac{1}{6^n} + \frac{1}{7^n} \pm \text{\&c.}$$

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{\&c.}$$

Si quidem n sit numerus par, atque summa erit $= A\varpi^n$ existente A numero rationali.

171. Præterea vero expressiones §. 164 exhibitæ simili mo-

do Series notatu dignas suppeditabunt. Cum enim sit $cos. \frac{1}{2} v + tang. \frac{1}{2} g \cdot sin. \frac{1}{2} v = (1 + \frac{v}{\varpi - g})(1 - \frac{v}{\varpi + g})(1 + \frac{v}{3\varpi - g})$ &c., si ponamus $v = \frac{x}{n} \varpi$ & $g = \frac{m}{n} \pi$ erit $(1 + \frac{x}{n-m})(1 - \frac{x}{n+m})(1 + \frac{x}{3n-m})(1 - \frac{x}{3n+m})(1 + \frac{x}{5n-m})(1 - \frac{x}{5n+m})$ &c. $= cos. \frac{x\varpi}{2n} + tang. \frac{m\varpi}{2n} \cdot sin. \frac{x\varpi}{2n} = 1 + \frac{\varpi x}{2n} tang. \frac{m\varpi}{2n} - \frac{\varpi\varpi xx}{2.4nn} - \frac{\varpi^3 x^3}{2.4.6n^3} tang. \frac{m\varpi}{2n} + \frac{\varpi^4 x^4}{2.4.6.8n^4} +$ &c.. Hæc expressio infinita cum §. 165 collata dabit hos valores $A = \frac{\varpi}{2n} tang. \frac{m\varpi}{2n}$; $B = \frac{-\varpi\varpi}{2.4nn}$; $C = \frac{-\varpi^3}{2.4.6n^3} \cdot tang. \frac{m\varpi}{2n}$, $D = \frac{\varpi^4}{2.4.6.8n^4}$; $E = \frac{\varpi^5}{2.4.6.8.10n^5} \cdot tang. \frac{m\varpi}{2n}$ &c. Tum vero erit $\alpha = \frac{1}{n-m}$; $\beta = -\frac{1}{n+m}$; $\gamma = \frac{1}{3n-m}$; $\delta = -\frac{1}{3n+m}$; $\varepsilon = \frac{1}{5n-m}$; $\zeta = -\frac{1}{5n+m}$ &c.

172. Hinc ergo ad normam §. 166 sequentes Series exorientur.

$$P = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \frac{1}{5n-m} - \frac{1}{5n+m} + \&c.$$

$$Q = \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(3n-m)^2} + \frac{1}{(3n+m)^2} + \frac{1}{(5n-m)^2} + \&c.$$

$$R = \frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} + \frac{1}{(3n-m)^3} - \frac{1}{(3n+m)^3} + \frac{1}{(5n-m)^3} - \&c.$$

$S =$

$S = \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(3n-m)^4} + \frac{1}{(3n+m)^4} + \frac{1}{(5n-m)^4} + \&c.$

$T = \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} + \frac{1}{(3n-m)^5} - \frac{1}{(3n+m)^5} + \frac{1}{(5n-m)^5} - \&c.$

$V = \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(3n-m)^6} + \frac{1}{(3n+m)^6} + \frac{1}{(5n-m)^6} + \&c.$

&c.

Posito autem *tang.* $\frac{m\varpi}{2n} = k$ erit, uti ostendimus,

$P = A = \frac{k\varpi}{2n} = \frac{k\pi}{2n}$

$Q = \frac{(kk+1)\pi\pi}{4nn} = \frac{(2kk+2)\pi^2}{2.4nn}$

$R = \frac{(k^3+k)\pi^3}{8n^3} = \frac{(6k^3+6k)\pi^3}{2.4.6.n^3}$

$S = \frac{(3k^4+4kk+1)\pi^4}{48n^4} = \frac{(24k^4+32k^2+8)\pi^4}{2.4.6.8n^4}$

$T = \frac{(3k^5+5k^3+2k)\pi^5}{96n^5} = \frac{(120k^5+200k^3+80k)\pi^5}{2.4.6.8.10n^5}$

173. Pari modo ultima forma §. 164; $cos.\frac{1}{2}v + cot.\frac{1}{2}g \times sin.\frac{1}{2}v = (1+\frac{v}{g})(1-\frac{v}{2\pi-g})(1+\frac{v}{2\pi+g})(1-\frac{v}{4\pi-g})(1+\frac{v}{4\pi+g})$ &c. Si ponamus $v = \frac{x}{n}\pi$, $g = \frac{m}{n}\pi$, & *tang.* $\frac{m\pi}{2n} = k$, ut sit $cot.\frac{1}{2}g = \frac{1}{k}$, dabit $cos.\frac{\pi x}{2n} + \frac{1}{k} \times sin.\frac{\pi x}{2n} = 1 + \frac{\pi x}{2nk} - \frac{\pi\pi xx}{2.4nn} - \frac{\pi^3x^3}{2.4.6n^3k} + \frac{\pi^4x^4}{2.4.6.8n^4} + \frac{\pi^5x^5}{2.4.6.8.10n^5k} - \&c. = (1+\frac{x}{m})(1-\frac{x}{2n-m})(1+\frac{x}{2n+m})$

$(1-$

LIB. I. $(1-\frac{x}{4n-m})(1+\frac{x}{4n+m})$ &c.. Comparatione ergo cum forma generali (§. 165) instituta erit $A=\frac{\pi}{2nk}$; $B=\frac{-\pi\pi}{2.4n^2}$; $C=\frac{-\pi^3}{2.4.6n^3k}$; $D=\frac{\pi^4}{2.4.6.8n^4}$; $E=\frac{\pi^5}{2.4.6.8.10n^5k}$; &c.; ex Factoribus vero habebitur $\alpha=\frac{1}{m}$; $\beta=\frac{-1}{2n-m}$; $\gamma=\frac{1}{2n+m}$; $\delta=\frac{-1}{4n-m}$; $\epsilon=\frac{1}{4n+m}$ &c.

174. Hinc ergo ad normam §. 166. sequentes Series formabuntur, earumque summæ assignabuntur

$$P=\frac{1}{m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{4n-m}+\frac{1}{4n+m}-\text{\&c.}$$

$$Q=\frac{1}{m^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+\frac{1}{(4n-m)^2}+\frac{1}{(4n+m)^2}+\text{\&c.}$$

$$R=\frac{1}{m^3}-\frac{1}{(2n-m)^3}+\frac{1}{(2n+m)^3}-\frac{1}{(4n-m)^3}+\frac{1}{(4n+m)^3}-\text{\&c.}$$

$$S=\frac{1}{m^4}+\frac{1}{(2n-m)^4}+\frac{1}{(2n+m)^4}+\frac{1}{(4n-m)^4}+\frac{1}{(4n+m)^4}+\text{\&c.}$$

$$T=\frac{1}{m^5}-\frac{1}{(2n-m)^5}+\frac{1}{(2n+m)^5}-\frac{1}{(4n-m)^5}+\frac{1}{(4n+m)^5}-\text{\&c.}$$

&c.

Hæ

Hæ autem ſummæ P, Q, R, S, &c. ita ſe habebunt CAP. X.

$$P = A = \frac{\pi}{2nk} = \frac{1\pi}{2nk}$$

$$Q = \frac{(kk+1)\pi\pi}{4nnkk} = \frac{(2+2kk)\pi^2}{2.4n^2k^2}$$

$$R = \frac{(kk+1)\pi^3}{8n^3k^3} = \frac{(6+6kk)\pi^3}{2.4.6n^3k^3}$$

$$S = \frac{(k^4+4kk+3)\pi^4}{48n^4k^4} = \frac{(24+32kk+3k^4)\pi^4}{2.4.6.8n^4k^4}$$

$$T = \frac{(2k^4+5kk+3)\pi^5}{96n^5k^5} = \frac{(120+200kk+80k^4)\pi^5}{2.4.6.8.10n^5k^5}$$

$$V = \frac{(2k^6+17k^4+30k^2+15)\pi^6}{960n^6k^6} = \frac{(720+1440kk+816k^4+96k^6)\pi^6}{2.4.6.8.10.12n^6k^6}$$

&c.

175. Series iſtæ generales merentur ut caſus quoſdam particulares inde derivemus, qui prodibunt ſi rationem m ad n in numeris determinemus. Sit igitur primum $m = 1$ & $n = 2$, fiet $k = tang.\ \frac{\varpi}{4} = tang.\ 45° = 1$, atque ambæ Serierum claſſes inter ſe congruent. Erit ergo

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$$

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \&c.$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.$$

$$\frac{5\pi^5}{1536} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \&c.$$

$$\frac{\pi^6}{960} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.$$

&c.

Harum Serierum primam jam ſupra (§. 140) elicuimus, reliquarum illæ, quæ pares habent Dignitates, modo ante (§. 169)

LIB. I. sunt erutæ; ceteræ, in quibus Exponentes sunt numeri impares, hic primum occurrunt. Constat ergo omnium quoque istarum Serierum:

$$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} - \&c.$$

summas per valorem ipsius π assignari posse.

176. Sit nunc $m = 1$, $n = 3$; erit $k = tang. \frac{\pi}{6} = tang.\ 30^\circ = \frac{1}{\sqrt{3}}$; atque Series §. 172 abibunt in has

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \&c$$

$$\frac{\pi\pi}{27} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{16^2} + \&c.$$

$$\frac{\pi^3}{162\sqrt{3}} = \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \frac{1}{16^3} + \&c.$$

&c., sive

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \&c.$$

$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c.$$

$$\frac{4\pi^3}{81\sqrt{3}} = 1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \&c.$$

&c.

in his Seriebus desunt omnes numeri per ternarium divisibiles: hinc pares dimensiones ex jam inventis deducentur hoc modo. Cum sit

$$\frac{\pi\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c., \text{ erit}$$

$$\frac{\pi\pi}{6.9} = \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \&c. = \frac{\pi\pi}{54},$$

quæ posterior Series continens omnes numeros per ternarium divisi-

divisibiles, si subtrahatur a priore, remanebunt omnes numeri non divisibiles per 3: sicque erit $\frac{8\pi\pi}{54} = \frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.$, uti invenimus.

177. Eadem hypothesis $m = 1$, $n = 3$, & $k = \frac{1}{\sqrt{3}}$, ad §. 174. accommodata has præbebit summationes

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \&c.$$
$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \&c.$$
$$\frac{\pi^3}{18\sqrt{3}} = 1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} - \&c.$$
&c.

in quarum denominatoribus numeri tantum impares occurrunt exceptis iis, qui per ternarium sunt divisibiles. Ceterum pares dimensiones ex jam cognitis deduci possunt, cum enim sit

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.,\ \text{erit}$$
$$\frac{\pi\pi}{8.\,9} = \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \&c. = \frac{\pi\pi}{72}$$

quæ Series, omnes numeros impares per 3 divisibiles continens, si subtrahatur a superiore, relinquet Seriem quadratorum numerorum imparium per 3 non divisibilium, eritque

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \&c.$$

178. Si Series in §. §. 172. & 174 inventæ vel addantur vel subtrahantur, obtinebuntur aliæ Series notatu dignæ. Erit scilicet $\frac{k\pi}{2n} + \frac{\pi}{2nk} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m}$

LIB. I. $\frac{1}{2n+m} + \&c. = \frac{(kk+1)\pi}{2nk}$: at eſt $k = tang. \frac{m\varpi}{2n} = \frac{ſin. \frac{m\pi}{2n}}{coſ. \frac{m\pi}{2n}}$, & $1 + kk = \frac{1}{(coſ. \frac{m\pi}{2n})^2}$, unde $\frac{2k}{1+kk} = 2 ſin. \frac{m\pi}{2n} \times coſ. \frac{m\pi}{2n} = ſin. \frac{m\pi}{n}$, quo valore ſubſtituto habebimus

$$\frac{\pi}{n\, ſin. \frac{m\varpi}{n}} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} - \&c.$$

Simili modo per ſubtractionem erit $\frac{\varpi}{2nk} - \frac{k\varpi}{2n} = \frac{(1-kk)\varpi}{2nk} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m} - \&c.$, at eſt $\frac{2k}{1-kk} = tang.\ 2. \frac{m\varpi}{2n} = tang. \frac{m\varpi}{n} = \frac{ſin. \frac{m\varpi}{n}}{coſ. \frac{m\varpi}{n}}$, hinc erit

$$\frac{\varpi\, coſ. \frac{m\varpi}{n}}{n\, ſin. \frac{m\varpi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \&c.$$

Series Quadratorum & altiorum Poteſtatum hinc ortæ facilius per differentiationem hinc deducentur infra.

179. Quoniam caſus, quibus $m = 1$ & $n = 2$ vel 3, jam evolvimus, ponamus $m = 1$ & $n = 4$; erit $ſin. \frac{m\pi}{n} = ſin. \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ & $coſ. \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Hinc itaque habebitur

$$\frac{\varpi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \&c.$$

&

& $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} +$ &c.. Sit $m = 1$ & $n = 8$, erit $\frac{m\varpi}{n} = \frac{\pi}{8}$ & *fin.* $\frac{\varpi}{8} = \surd(\frac{1}{2} - \frac{1}{2\surd 2})$ & *cof.* $\frac{\pi}{8} = \surd(\frac{1}{2} + \frac{1}{2\surd 2})$ & $\frac{cof.\ \frac{\pi}{8}}{fin.\ \frac{\pi}{8}} = 1 + \surd 2$. Hinc itaque erit

CAP. X.

$$\frac{\varpi}{4\surd(2-\surd 2)} = 1 + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \text{\&c.}$$

$$\frac{\varpi}{8(\surd 2 - 1)} = 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \text{\&c.}$$

Sit nunc $m = 3$ & $n = 8$, erit $\frac{m\varpi}{n} = \frac{3\varpi}{8}$ & *fin.* $\frac{3\varpi}{8} = \surd(\frac{1}{2} + \frac{1}{2\surd 2})$, & *cof.* $\frac{3\pi}{8} = \surd(\frac{1}{2} - \frac{1}{2\surd 2})$, unde $\frac{cof.\ \frac{3}{8}\varpi}{fin.\ \frac{3}{8}\varpi} = \frac{1}{\surd 2 + 1}$; ac prodibunt hæ Series

$$\frac{\varpi}{4\surd(2+\surd 2)} = \frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} - \text{\&c.}$$

$$\frac{\varpi}{8(\surd 2 + 1)} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \text{\&c.}$$

180. Ex his Seriebus per combinationem naſcuntur:

$$\frac{\pi\surd(2+\surd 2)}{4} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \text{\&c.}$$

$$\frac{\pi\surd(2-\surd 2)}{4} = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \text{\&c.}$$

$$\frac{\pi(\surd(4+2\surd 2)+\surd 2 - 1)}{8} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \text{\&c.}$$

$$\frac{\pi(\sqrt{(4+2\sqrt{2})}-\sqrt{2}+1)}{8} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \&c.$$

$$\frac{\pi(\sqrt{2}+1+\sqrt{(4-2\sqrt{2})})}{8} = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

$$\frac{\pi(\sqrt{2}+1-\sqrt{(4-2\sqrt{2})})}{8} = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} - \&c.$$

Simili modo, ponendo $n = 16$ & m vel 1 vel 3 vel 5 vel 7, ulterius progredi licet, hocque modo ſummæ reperientur Serierum 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, &c., in quibus ſignorum + & — viciſſitudines alias leges ſequantur.

181. Si in Seriebus §. 178. inventis bini termini in unam ſummam colligantur, erit

$$\frac{\pi}{n\,\mathrm{ſin.}\frac{m\pi}{n}} = \frac{1}{m} + \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} + \frac{2m}{9nn-mm} - \frac{2m}{16nn-mm} + \&c.$$

ideoque

$$\frac{1}{nn-mm} - \frac{1}{4nn-mm} + \frac{1}{9nn-mm} - \&c. = \frac{\pi}{2mn.\,\mathrm{ſin.}\frac{m\pi}{n}} - \frac{1}{2mm}.$$

Altera vero Series dabit

$$\frac{\pi}{n\,\mathrm{tang.}\frac{m}{n}\pi} = \frac{1}{m} - \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} - \frac{2m}{9nn-mm} - \&c.$$

hincque

hincque CAP X.

$$\frac{1}{nn-mm}+\frac{1}{4nn-mm}+\frac{1}{9nn-mm}+\&c.=\frac{1}{2mm}-\frac{\pi}{2mn\,tang.\frac{m}{n}\pi}.$$

Ex his autem conjunctis nascitur hæc

$$\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\&c.=\frac{\pi\,tang.\frac{m}{2n}\pi}{4mn}.$$

Si in hac Serie sit $n=1$ & m numerus par quicunque $=2k$, ob $tang.\,k\pi=0$, erit semper, nisi sit $k=0$,

$$\frac{1}{1-4kk}+\frac{1}{9-4kk}+\frac{1}{25-4kk}+\frac{1}{49-4kk}+\&c.=0,$$

sin autem in illa Serie fiat $n=2$ & m fuerit numerus quicunque impar $=2k+1$, ob $\frac{1}{tang.\frac{m\pi}{n}}=0$, erit $\frac{1}{4-(2k+1)^2}+$

$$\frac{1}{16-(2k+1)^2}+\frac{1}{36-(2k+1)^2}+\&c.=\frac{1}{2(2k+1)^2}.$$

182. Multiplicentur Series inventæ per nn sitque $\frac{m}{n}=p$, habebuntur istæ formæ

$$\frac{1}{1-pp}-\frac{1}{4-pp}+\frac{1}{9-pp}-\frac{1}{16-pp}+\&c.=\frac{\pi}{2p\,sin.\,p\pi}-\frac{1}{2pp}.$$

$$\frac{1}{1-pp}+\frac{1}{4-pp}+\frac{1}{9-pp}+\frac{1}{16-pp}+\&c.=\frac{1}{2pp}-\frac{\pi}{2p\,sin.\,p\pi}.$$

Sit $pp=a$, atque nascentur hæ Series

$$\frac{1}{1-a}-\frac{1}{4-a}+\frac{1}{9-a}-\frac{1}{16-a}+\&c.=\frac{\pi\sqrt{a}}{2a\,sin.\,\pi\sqrt{a}}-\frac{1}{2a}$$

$$\frac{1}{1-a}+\frac{1}{4-a}+\frac{1}{9-a}+\frac{1}{16-a}+\&c.=\frac{1}{2a}-\frac{\pi\sqrt{a}}{2a\,tang.\pi\sqrt{a}}$$

Dummodo ergo a non fuerit numerus negativus nec quadratus integer, summa harum Serierum per Circulum exhiberi poterit.

183

Lib. I.

183. Per reductionem autem exponentialium imaginariorum ad Sinus & Cosinus Arcuum circularium supra traditam poterimus quoque summas harum Serierum assignare si a sit numerus negativus. Cum enim sit $e^{x\sqrt{-1}} = cos.\, x + \sqrt{-1} \times sin.\, x$ & $e^{-x\sqrt{-1}} = cos.\, x - \sqrt{-1}.\, sin.\, x$, erit vicissim, posito $y\sqrt{-1}$ loco x; $cos.\, y\sqrt{-1} = \frac{e^{-y} + e^{y}}{2}$ & $sin.\, y\sqrt{-1} = \frac{e^{-y} - e^{y}}{2\sqrt{-1}}$. Quod si ergo $a = -b$ & $y = \varpi\sqrt{b}$, erit $cos.\, \varpi\sqrt{-b} = \frac{e^{-\varpi\sqrt{b}} + e^{\varpi\sqrt{b}}}{2}$ & $sin.\, \varpi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{2\sqrt{-1}}$; ideoque $tang.\, \pi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{(e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}})\sqrt{-1}}$. Hinc erit $\frac{\pi\sqrt{-b}}{sin.\, \varpi\sqrt{-b}} = \frac{-2\varpi\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}$; & $\frac{\varpi\sqrt{-b}}{tang.\, \pi\sqrt{-b}} = \frac{(e^{-\varpi\sqrt{b}} + e^{\varpi\sqrt{b}})\pi\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}$. His ergo notatis, erit

$$\frac{1}{1+b} - \frac{1}{4+b} + \frac{1}{9+b} - \frac{1}{16+b} + \&c. = \frac{1}{2b} - \frac{\pi\sqrt{b}}{(e^{\pi\sqrt{b}} - e^{-\pi\sqrt{b}})b}$$

$$\frac{1}{1+b} + \frac{1}{4+b} + \frac{1}{9+b} + \frac{1}{16+b} + \&c. = \frac{(e^{\pi\sqrt{b}} + e^{-\pi\sqrt{b}})\varpi\sqrt{b}}{2b(e^{\pi\sqrt{b}} - e^{-\varpi\sqrt{b}})} - \frac{1}{2b}.$$

Eædem hæ Series deduci possunt ex §. 162. adhibendo eandem methodum, qua in

in hoc capite ſum uſus. Quoniam vero hoc pacto reductio Sinuum & Coſinuum Arcuum imaginariorum ad quantitates exponentiales reales, non mediocriter illuſtratur, hanc explicationem alteri præferendam duxi.

CAPUT XI.

De aliis Arcuum atque Sinuum expreſſionibus infinitis.

184. QUoniam ſupra (158.), denotante z Arcum Circuli quemcunque, vidimus eſſe $\sin. z = z\left(1-\frac{zz}{\pi\pi}\right)\left(1-\frac{zz}{4\pi\pi}\right)\left(1-\frac{zz}{9\pi\pi}\right)\left(1-\frac{zz}{16\pi\pi}\right)$ &c., & $\cos. z = \left(1-\frac{4zz}{\pi\pi}\right)\left(1-\frac{4zz}{9\pi\pi}\right)\left(1-\frac{4zz}{25\pi\pi}\right)\left(1-\frac{4zz}{49\pi\pi}\right)$ &c., ponamus eſſe Arcum $z=\frac{m\pi}{n}$, erit $\sin.\frac{m\pi}{n}=\frac{m\pi}{n}\left(1-\frac{mm}{nn}\right)\left(1-\frac{mm}{4nn}\right)\left(1-\frac{mm}{9nn}\right)\left(1-\frac{mm}{16nn}\right)$ &c., & $\cos.\frac{m}{n}\pi = \left(1-\frac{4mm}{nn}\right)\left(1-\frac{4mm}{9nn}\right)\left(1-\frac{4mm}{25nn}\right)\left(1-\frac{4mm}{49nn}\right)$ &c..
Vel ponatur $2n$ loco n, ut prodeant hæ expreſſiones

$$\sin.\frac{m\pi}{2n}=\frac{m\pi}{2n}.\left(\frac{4nn-mm}{4nn}.\right)\left(\frac{16nn-mm}{16nn}.\right)\left(\frac{36nn-mm}{36nn}\right)\text{ \&c.}$$

$$\cos.\frac{m\pi}{2n}=\left(\frac{nn-mm}{nn}.\right)\left(\frac{9nn-mm}{9nn}.\right)\left(\frac{25nn-mm}{25nn}.\right)\left(\frac{49nn-mm}{49nn}\right)\text{\&c.},$$

quæ, in Factores ſimplices reſolutæ, dant

$$\sin.\frac{m\pi}{2n}=\frac{m\pi}{2n}.\left(\frac{2n-m}{2n}.\right)\left(\frac{2n+m}{2n}.\right)\left(\frac{4n-m}{4n}.\right)\left(\frac{4n+m}{4n}.\right)\left(\frac{6n-m}{6n}\right)\text{ \&c.}$$

LIB. I. $cos. \frac{m\pi}{2n} = \left(\frac{n-m}{n}\right)\left(\frac{n+m}{n}\right)\left(\frac{3n-m}{3n}\right)\left(\frac{3n+m}{3n}\right)\left(\frac{5n-m}{5n}\right)$

$\left(\frac{5n+m}{5n}\right)$ &c.

Ponatur $n-m$ loco m, quia est $fin. \frac{(n-m)\pi}{2n} = cos. \frac{m\pi}{2n}$ & $cos. \frac{(n-m)\pi}{2n} = fin. \frac{m\pi}{2n}$, provenient hæ expressiones.

$$cos. \frac{m\pi}{2n} = \left(\frac{(n-m)\pi}{2n}\right)\left(\frac{n+m}{2n}\right)\left(\frac{3n-m}{2n}\right)\left(\frac{3n+m}{4n}\right)$$

$$\left(\frac{5n-m}{4n}\right)\left(\frac{5n+m}{6n}\right) \text{ \&c.}$$

$$fin. \frac{m\pi}{2n} = \frac{m}{n}\left(\frac{2n-m}{n}\right)\left(\frac{2n+m}{3n}\right)\left(\frac{4n-m}{3n}\right)\left(\frac{4n+m}{5n}\right)$$

$$\left(\frac{6n-m}{5n}\right) \text{ \&c.}$$

185. Cum igitur pro Sinu & Cosinu Anguli $\frac{m\pi}{2n}$ binæ habeantur expressiones, si eæ inter se comparentur dividendo, erit $1 = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{7}{8} \cdot \frac{9}{8} \cdot$ &c., ideoque $\frac{\pi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12.\ 12.}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11.\ 13.}$ &c., quæ est expressio pro Peripheria Circuli, quam WALLISIUS invenit in *Arithmetica infinitorum*. Similes autem huic innumeras expressiones exhibere licet ope primæ expressionis pro Sinu; ex ea enim deducitur fore:

$$\frac{\pi}{2} = \frac{n}{m}.fin. \frac{m\pi}{2n}\left(\frac{2n}{2n-m}\right)\left(\frac{2n}{2n+m}\right)\left(\frac{4n}{4n-m}\right)\left(\frac{4n}{4n+m}\right)\left(\frac{6n}{6n-m}\right) \text{ \&c.},$$

quæ, posito $\frac{m}{n} = 1$, præbet illam ipsam WALLISII formulam.

Sit

Sit ergo $\frac{m}{n} = \frac{1}{2}$, ob $\mathit{sin.}\frac{1}{4}\pi = \frac{1}{\sqrt{2}}$, erit

$$\frac{\pi}{2} = \frac{\sqrt{2}}{1} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{16}{15} \cdot \frac{16}{17} \cdot \&c.$$

Sit $\frac{m}{n} = \frac{1}{3}$, ob $\mathit{sin.}\frac{1}{6}\pi = \frac{1}{2}$, erit

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{18}{17} \cdot \frac{18}{19} \cdot \frac{24}{23}\ \&c..$$

Quod si Expressio *Wallisiana* dividatur per illam ubi $\frac{m}{n} = \frac{1}{2}$, erit $\sqrt{2} = \frac{2.2.6.6.10.10.14.14.18.18.}{1.3.5.7.\ 9.11.13.15.17.19.}$ &c.

186. Quoniam Tangens cujusque Anguli æquatur Sinui per Cosinum diviso, Tangens quoque per hujusmodi Factores infinitos exprimi poterit. Quod si autem prima Sinus expressio dividatur per alteram Cosinus expressionem, erit

$$\mathit{tang.}\frac{m\pi}{2n} = \frac{m}{n-m}\left(\frac{2n-m}{n+m}\right)\left(\frac{2n+m}{3n-m}\right)\left(\frac{4n-m}{3n+m}\right)\left(\frac{4n+m}{5n-m}\right)\ \&c.,$$

&

$$\mathit{cot.}\frac{m\pi}{2n} = \frac{n-m}{m}\left(\frac{n+m}{2n-m}\right)\left(\frac{3n-m}{2n+m}\right)\left(\frac{3n+m}{4n-m}\right)\left(\frac{5n-m}{4n+m}\right)\ \&c..$$

Simili modo autem Secantes & Cosecantes exprimentur

$$\mathit{sec.}\frac{m\pi}{2n} = \left(\frac{n}{n-m}\right)\left(\frac{n}{n+m}\right)\left(\frac{3n}{3n-m}\right)\left(\frac{3n}{3n+m}\right)\left(\frac{5n}{5n-m}\right)\left(\frac{5n}{5n+m}\right)\ \&c.$$

$$\mathit{cosec.}\frac{m\pi}{2n} = \frac{n}{m}\left(\frac{n}{2n-m}\right)\left(\frac{3n}{2n+m}\right)\left(\frac{3n}{4n-m}\right)\left(\frac{5n}{4n+m}\right)\left(\frac{5n}{6n-m}\right)\ \&c.$$

Sin autem alteræ Sinuum & Cosinuum formulæ combinentur, erit

Lib. I.

$$tang.\ \frac{m\pi}{2n} = \frac{\pi}{2}\cdot\frac{m}{n-m}\cdot\frac{1(2n-m)}{2(n+m)}\cdot\frac{3(2n+m)}{2(3n-m)}\cdot\frac{3(4n-m)}{4(3n+m)}.\&c.$$

$$cot.\ \frac{m\pi}{2n} = \frac{\pi}{2}\cdot\frac{n-m}{m}\cdot\frac{1(n+m)}{2(2n-m)}\cdot\frac{3(3n-m)}{2(2n+m)}\cdot\frac{3(3n+m)}{4(4n-m)}.\&c.$$

$$sec.\ \frac{m\pi}{2n} = \frac{2}{\pi}\cdot\frac{n}{n-m}\cdot\frac{2n}{n+m}\cdot\frac{2n}{3n-m}\cdot\frac{4n}{3n+m}\cdot\frac{4n}{5n-m}.\&c.$$

$$cosec.\ \frac{m\pi}{2n} = \frac{2}{\pi}\cdot\frac{n}{m}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}.\&c.$$

187. Si loco m scribatur k, similique modo Anguli $\frac{k\pi}{2n}$ Sinus & Cosinus definiantur, ac per has expressiones illæ priores dividantur, prodibunt istæ formulæ

$$\frac{sin.\ \frac{m\pi}{2n}}{sin.\ \frac{k\pi}{2n}} = \frac{m}{k}\cdot\frac{2n-m}{2n-k}\cdot\frac{2n+m}{2n+k}\cdot\frac{4n-m}{4n-k}\cdot\frac{4n+m}{4n+k}\cdot\ \&c.$$

$$\frac{sin.\ \frac{m\pi}{2n}}{cos.\ \frac{k\pi}{2n}} = \frac{m}{n-k}\cdot\frac{2n-m}{n+k}\cdot\frac{2n+m}{3n-k}\cdot\frac{4n-m}{3n+k}\cdot\frac{4n+m}{5n-k}\cdot\ \&c.$$

$$\frac{cos.\ \frac{m\pi}{2n}}{cos.\ \frac{k\pi}{2n}} = \left(\frac{n-m}{n-k}\right)\left(\frac{n+m}{n+k}\right)\left(\frac{3n-m}{3n-k}\right)\left(\frac{3n+m}{3n+k}\right)\left(\frac{5n-m}{5n-k}\right).\ \&c.$$

$$\frac{cos.\ \frac{m\pi}{2n}}{sin.\ \frac{k\pi}{2n}} = \left(\frac{n-m}{k}\right)\left(\frac{n+m}{2n-k}\right)\left(\frac{3n-m}{2n+k}\right)\left(\frac{3n+m}{4n-k}\right)\left(\frac{5n-m}{4n+k}\right).\ \&c.$$

Sumto ergo pro $\frac{k\pi}{2n}$ ejusmodi Angulo cujus Sinus & Cosinus dentur, per hos licebit alius cujuscunque Anguli $\frac{m\pi}{2n}$ Sinum & Cosinum determinare.

188. Vicissim igitur hujusmodi expressionum, quæ ex Factoribus

ctoribus infinitis constant, valores veri vel per Circuli Peripheriam, vel per Sinus & Cosinus Angulorum datorum assignari possunt, quod ipsum non parvi est momenti, cum etiamnunc aliæ methodi non constent, quarum ope hujusmodi productorum infinitorum valores exhiberi queant. Ceterum vero hujusmodi expressiones parum utilitatis afferunt, ad valores cum ipsius π tum Sinuum Cosinuumve Angulorum $\frac{m\pi}{2n}$ per approximationem eruendos. Quanquam enim isti Factores $\frac{\pi}{2} = 2(1 - \frac{1}{9})(1 - \frac{1}{25})(1 - \frac{1}{49})$ &c., in fractionibus decimalibus non difficulter in se multiplicantur, tamen nimis multi termini in computum duci deberent, si valorem ipsius π ad decem tantum figuras justum invenire vellemus. Cap. XI.

189. Præcipuus autem usus hujusmodi expressionum, etsi infinitarum, in inventione Logarithmorum versatur, in quo negotio Factorum utilitas tanta est, ut sine illis Logarithmorum supputatio esset difficillima. Ac primo quidem, cum sit $\pi = 4(1 - \frac{1}{9})(1 - \frac{1}{25})(1 - \frac{1}{49})$ &c., erit, sumendis Logarithmis, $l\pi = l4 + l(1 - \frac{1}{9}) + l(1 - \frac{1}{25}) + l(1 - \frac{1}{49}) +$ &c., vel $l\pi = l2 - l(1 - \frac{1}{4}) - l(1 - \frac{1}{16}) - l(1 - \frac{1}{36})$ — &c., sive Logarithmi communes sive hyperbolici sumantur. Quoniam vero ex Logarithmis hyperbolicis vulgares facile reperiuntur, insigne compendium adhiberi poterit ad Logarithmum hyperbolicum ipsius π inveniendum.

190. Cum igitur, Logarithmis hyperbolicis sumendis, sit $l(1 - x) = -x - \frac{xx}{2} - \frac{x^3}{3} - \frac{x^4}{4} -$ &c., si hoc modo singuli termini evolvantur, erit

$$l\pi = l4 \begin{cases} -\frac{1}{9} - \frac{1}{2.\,9^2} - \frac{1}{3.\,9^3} - \frac{1}{4.\,9^4} - \&c. \\ -\frac{1}{25} - \frac{1}{2.\,25^2} - \frac{1}{3.\,25^3} - \frac{1}{4.\,25^4} - \&c. \\ -\frac{1}{49} - \frac{1}{2.\,49^2} - \frac{1}{3.\,49^3} - \frac{1}{4.\,49^4} - \&c. \\ \qquad\qquad \&c. \end{cases}$$

In his Seriebus numero infinitis verticaliter defcendendo ejusmodi prodeunt Series, quarum fummas fupra jam invenimus, quare fi brevitatis gratia ponamus

$$A = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$
$$B = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.$$
$$C = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.$$
$$D = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \&c.$$
$$\&c.$$

erit $l\pi = l4 - (A - 1) - \frac{1}{2}(B - 1) - \frac{1}{3}(C - 1) - \frac{1}{4}(D - 1) - \&c.$

Eft vero, fummis fupra inventis proxime exprimendis,

$$\begin{aligned} A &= 1,\,23370055013616982735431 \\ B &= 1,\,01467803160419205454625 \\ C &= 1,\,00144707664094212190647 \\ D &= 1,\,00015517902529611930298 \\ E &= 1,\,00001704136304482550816 \\ F &= 1,\,00000188584858311957590 \\ G &= 1,\,00000020924051921150010 \\ H &= 1,\,00000002323715737915670 \end{aligned}$$

$I =$

I	$=$	1, 00000000258143755665977
K	$=$	1, 00000000028680769745558
L	$=$	1, 00000000003186677514044
M	$=$	1, 00000000000354072294392
N	$=$	1, 00000000000039341246691
O	$=$	1, 00000000000004371244859
P	$=$	1, 00000000000000485693682
Q	$=$	1, 00000000000000053965957
R	$=$	1, 00000000000000005996217
S	$=$	1, 00000000000000000666246
T	$=$	1, 00000000000000000074027
V	$=$	1, 00000000000000000008225
W	$=$	1, 00000000000000000000913
X	$=$	1, 00000000000000000000101

Hinc ſine tædioſo calculo reperitur Logarithmus hyperbolicus ipſius $\pi = 1, 1447298858494001741434 2$, qui ſi multiplicetur per $0, 43429$ &c., prodit Logarithmus vulgaris ipſius $\pi = 0, 497149872694133854351 26$.

191. Quia porro tam Sinum quam Coſinum Anguli $\frac{m\pi}{2n}$ expreſſum habemus per Factores numero infinitos, utriuſque Logarithmum commode exprimere poterimus. Erit autem ex formulis primo inventis

$$l\,\mathit{ſin}.\frac{m\pi}{2n} = l\,\pi + l\,\frac{m}{2n} + l\left(1 - \frac{mm}{4nn}\right) + l\left(1 - \frac{mm}{16nn}\right) + l\left(1 - \frac{mm}{36nn}\right)\ \&c.$$

$$l\,\mathit{coſ}.\frac{m\pi}{2n} = l\left(1 - \frac{mm}{nn}\right) + l\left(1 - \frac{mm}{9nn}\right) + l\left(1 - \frac{mm}{25nn}\right) + l\left(1 - \frac{mm}{49nn}\right) + \&c.$$

Hinc primum Logarithmi hyperbolici, ut ante, per Series maxime convergentes facile exprimuntur. Ne autem præter neceſſi-

LIB. I. necessitatem Series infinitas multiplicemus, terminos priores actu in Logarithmis involutos relinquamus, eritque

$$l\sin.\frac{m\pi}{2n} = l\pi + lm + l(2n - m) + l(2n + m) - l8 - 3ln$$
$$- \frac{mm}{16nn} - \frac{m^4}{2.16^2n^4} - \frac{m^6}{3.16^3n^6} - \frac{m^8}{4.16^4n^8} - \&c.$$
$$- \frac{mm}{36nn} - \frac{m^4}{2.36^2n^4} - \frac{m^6}{3.36^3n^6} - \frac{m^8}{4.36^4n^8} - \&c.$$
$$- \frac{mm}{64nn} - \frac{m^4}{2.64^2n^4} - \frac{m^6}{3.64^3n^6} - \frac{m^8}{4.64^4n^8} - \&c.$$
$$\&c.$$

$$l\cos.\frac{m\pi}{2n} = l(n - m) + l(n + m) - 2\,ln$$
$$- \frac{mm}{9nn} - \frac{m^4}{2.\,9^2n^4} - \frac{m^6}{3.\,9^3n^6} - \frac{m^8}{4.\,9^4n^8} - \&c.$$
$$- \frac{mm}{25nn} - \frac{m^4}{2.25^2n^4} - \frac{m^6}{3.25^3n^6} - \frac{m^8}{4.25^4n^8} - \&c.$$
$$- \frac{mm}{49nn} - \frac{m^4}{2.49^2n^4} - \frac{m^6}{3.49^3n^6} - \frac{m^8}{4.49^4n^8} - \&c.$$
$$\&c.$$

192. Occurrunt ergo in his Seriebus singulæ Potestates pares ipsius $\frac{m}{n}$, quæ sunt multiplicatæ per Series, quarum summas jam supra assignavimus. Erit nempe

$$l\sin.\frac{m\varpi}{2n} = lm + l(2n - m) + l(2n + m) - 3\,ln + l\pi - l8$$
$$- \frac{mm}{nn}\left(\frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \&c.\right)$$
$$- \frac{m^4}{2n^4}\left(\frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \frac{1}{10^4} + \frac{1}{12^4} + \&c.\right)$$
$$- \frac{m^6}{3n^6}\left(\frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \frac{1}{10^6} + \frac{1}{12^6} + \&c.\right)$$
$$- \frac{m^8}{4n^8}\left(\frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \frac{1}{10^8} + \frac{1}{12^8} + \&c.\right)$$
$$\&c.$$

$$l\,cos.\,\frac{m\pi}{2n} = l(n-m) + l(n+m) - 2\,l\,n$$

$$-\frac{mm}{nn}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\&c.\right)$$
$$-\frac{m^4}{2n^4}\left(\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\&c.\right)$$
$$-\frac{m^6}{3n^6}\left(\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\&c.\right)$$
$$-\frac{m^8}{4n^8}\left(\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{9^8}+\&c.\right)$$
&c.

Serierum poſteriorum modo ante (§. 190) ſummæ ſunt exhibitæ; priores Series quidem ex his derivari poſſent, at, quo facilius ad uſum transferri queant, earum ſummas pariter hic adjiciam.

193. Quod ſi ergo, brevitatis gratia, ponamus

$$\alpha = \frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\&c.$$
$$\beta = \frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\&c.$$
$$\gamma = \frac{1}{2^6}+\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\&c.$$
$$\delta = \frac{1}{2^8}+\frac{1}{4^8}+\frac{1}{6^8}+\frac{1}{8^8}+\&c.$$
&c.

erunt ſummæ in numeris proxime expreſſæ hæ:

$$\alpha = 0,41123351671205660911810$$
$$\beta = 0,06764520210694613696975$$
$$\gamma = 0,01589598534350701780804$$
$$\delta = 0,00392217717264822007570$$
$$\varepsilon = 0,00097753376477325984898$$
$$\zeta = 0,00024420070472492872274$$

LIB. I.

η =	0,00006103889453949332915
θ =	0,00001525902225127269977
ι =	0,00000381471182744318008
κ =	0,00000095367522617534053
λ =	0,00000023841863595259154
μ =	0,00000005960464832831555
ν =	0,00000001490116141589813
ξ =	0,00000000372529031233986
o =	0,00000000093132257548284
π =	0,00000000023283064370807
ρ =	0,00000000005820766091685
σ =	0,00000000001455191522858
τ =	0,00000000000363797880710
υ =	0,00000000000090949470177
ϕ =	0,00000000000022737367544
χ =	0,00000000000005684341886
ψ =	0,00000000000001421085471
ω =	0,00000000000000355271367

reliquæ ſummæ in ratione quadrupla deſcreſcunt.

194. His ergo in ſubſidium vocatis, erit

$$l\,\mathit{ſin.}\,\frac{m\pi}{2n} = lm + l(2n-m) + l(2n+m) - 3ln + l\varpi - l8$$

$$-\frac{mm}{nn}\left(\alpha-\frac{1}{2^2}\right)-\frac{m^4}{2n^4}\left(\beta-\frac{1}{2^4}\right)-\frac{m^6}{3n^6}\left(\gamma-\frac{1}{2^6}\right)$$

$-$ &c.

$$l\,\mathit{coſ.}\,\frac{m\varpi}{2n} = l(n-m) + l(n+m) - 2ln$$

$$-\frac{mm}{nn}(A-1)-\frac{m^4}{2n^4}(B-1)-\frac{m^6}{3n^6}(C-1)-\text{\&c.}$$

quoniam igitur Logarithmi $l\varpi$ & $l8$ dantur, erit

Logarith-

CAP. XI.

Logarithmus hyperbolicus Sinus Anguli $\frac{m}{n} 90° =$

$$lm + l(2n - m) + l(2n + m) - 3ln$$

$$- \quad 0,93471165583043575410$$

$$- \frac{m^2}{n^2} \cdot 0,16123351671205660911$$

$$- \frac{m^4}{n^4} \cdot 0,00257260105347306848$$

$$- \frac{m^6}{n^6} \cdot 0,00009032844783567260$$

$$- \frac{m^8}{n^8} \cdot 0,00000398179316205501$$

$$- \frac{m^{10}}{n^{10}} \cdot 0,00000019425295465196$$

$$- \frac{m^{12}}{n^{12}} \cdot 0,00000001001328748812$$

$$- \frac{m^{14}}{n^{14}} \cdot 0,00000000053404135618$$

$$- \frac{m^{16}}{n^{16}} \cdot 0,00000000002914859658$$

$$- \frac{m^{18}}{n^{18}} \cdot 0,00000000000161797979$$

$$- \frac{m^{20}}{n^{20}} \cdot 0,00000000000009097690$$

$$- \frac{m^{22}}{n^{22}} \cdot 0,00000000000000516827$$

$$- \frac{m^{24}}{n^{24}} \cdot 0,00000000000000029607$$

$$- \frac{m^{26}}{n^{26}} \cdot 0,00000000000000001708$$

$$- \frac{m^{28}}{n^{28}} \cdot 0,00000000000000000099$$

$$- \frac{m^{30}}{n^{30}} \cdot 0,00000000000000000005$$

LIB. I.

At *Logarithmus hyperbolicus Cosinus Ang.* $\frac{m}{n} 90^\circ =$

$$l(n-m) + l(n+m) - 2\,ln$$

$$-\frac{m^{2}}{n^{2}}.\ 0,\ 23370055013616982735$$

$$-\frac{m^{4}}{n^{4}}.\ 0,\ 00733901580209602727$$

$$-\frac{m^{6}}{n^{6}}.\ 0,\ 00048235888031404063$$

$$-\frac{m^{8}}{n^{8}}.\ 0,\ 00003879475632402982$$

$$-\frac{m^{10}}{n^{10}}.\ 0,\ 00000340827260896510$$

$$-\frac{m^{12}}{n^{12}}.\ 0,\ 00000031430809718659$$

$$-\frac{m^{14}}{n^{14}}.\ 0,\ 00000002989150274450$$

$$-\frac{m^{16}}{n^{16}}.\ 0,\ 00000000290464467239$$

$$-\frac{m^{18}}{n^{18}}.\ 0,\ 00000000028682639518$$

$$-\frac{m^{20}}{n^{20}}.\ 0,\ 00000000002868076974$$

$$-\frac{m^{22}}{n^{22}}.\ 0,\ 00000000000289697956$$

$$-\frac{m^{24}}{n^{24}}.\ 0,\ 00000000000029506024$$

$$-\frac{m^{26}}{n^{26}}.\ 0,\ 00000000000003026249$$

$$-\frac{m^{28}}{n^{28}}.\ 0;\ 00000000000000312232$$

$$-\frac{m^{30}}{n^{30}}.\ 0,\ 00000000000000032379$$

$$-\frac{m^{32}}{n^{32}}.\ 0,\ 00000000000000003373$$

$$-\frac{m^{34}}{n^{34}}.\ 0,\ 00000000000000000352$$

$-\frac{m^{36}}{n^{36}}.$ 0, 0000000000000000037

$-\frac{m^{38}}{n^{38}}.$ 0, 0000000000000000004

195. Si isti Sinuum & Cosinuum Logarithmi hyperbolici multiplicentur per 0, 4342944819 &c., prodibunt eorundem Logarithmi vulgares ad Radium = 1 relati. Quoniam vero in Tabulis Logarithmus Sinus totius statui solet = 10, quo Logarithmi tabulares Sinuum & Cosinuum obtineantur, post multiplicationem addi debet 10. Hinc erit

Logarithmus tabularis Sinus Anguli $\frac{m}{n}\,90° =$
$$l\,m + l(2n - m) + l(2n + m) - 3\,ln$$

$+$	9, 594059885702190
$-\frac{m^2}{n^2}.$	0, 070022826605901
$-\frac{m^4}{n^4}.$	0, 001117266441661
$-\frac{m^6}{n^6}.$	0, 000039229146453
$-\frac{m^8}{n^8}.$	0, 000001729270798
$-\frac{m^{10}}{n^{10}}.$	0, 000000084362986
$-\frac{m^{12}}{n^{12}}.$	0, 000000004348715
$-\frac{m^{14}}{n^{14}}.$	0, 000000000231931
$-\frac{m^{16}}{n^{16}}.$	0, 000000000012659
$-\frac{m^{18}}{n^{18}}.$	0, 000000000000702
$-\frac{m^{20}}{n^{20}}.$	0, 000000000000039

Logarithmus tabularis Coſinus Anguli $\frac{m}{n}\, 90^\circ =$
$l(n-m) + l(n+m) - 2\,l\,n$

$+$		10, 000000000000000
$-$	$\frac{m^2}{n^2}.$	0, 101494859341892
$-$	$\frac{m^4}{n^4}.$	0, 003187294065451
$-$	$\frac{m^6}{n^6}.$	0, 000209485800017
$-$	$\frac{m^8}{n^8}.$	0, 000016848348597
$-$	$\frac{m^{10}}{n^{10}}.$	0, 000001480193986
$-$	$\frac{m^{12}}{n^{12}}.$	0, 000000136502272
$-$	$\frac{m^{14}}{n^{14}}.$	0, 000000012981715
$-$	$\frac{m^{16}}{n^{16}}.$	0, 000000001261471
$-$	$\frac{m^{18}}{n^{18}}.$	0, 000000000124567
$-$	$\frac{m^{20}}{n^{20}}.$	0, 000000000012456
$-$	$\frac{m^{22}}{n^{22}}.$	0, 000000000001258
$-$	$\frac{m^{24}}{n^{24}}.$	0, 000000000000128
$-$	$\frac{m^{26}}{n^{26}}.$	0, 000000000000013

196. Harum ergo formularum ope inveniri poſſunt Logarithmi Sinuum & Coſinuum quorumvis Angulorum tam hyperbolici quam vulgares, etiam ignoratis ipſis Sinibus & Coſinibus. Ex Logarithmis autem Sinuum & Coſinuum per ſolam ſubtractionem inveniuntur Logarithmi Tangentium, Cotangen-

tangentium, & Secantium, Cosecantiumque, quamobrem pro his peculiaribus formulis non erit opus. Ceterum notandum est numerorum m, n, $n-m$, $n+m$, &c. Logarithmos hyperbolicos accipi oportere, cum Logarithmi hyperbolici Sinuum Cosinuumque quæruntur, vulgares autem, cum tales ope posteriorum formularum sunt indagandi. Præterea $m : n$ denotat rationem, quam Angulus propositus habet ad Angulum rectum; sicque, cum Sinus Angulorum semirecto majorum æquentur Cosinibus Angulorum semirecto minorum ac vicissim, fractio $\frac{m}{n}$ nunquam major accipienda erit quam $\frac{1}{2}$, hancque ob rem termini illi multo magis convergent, ut semissis instituto sufficere possit.

197. Antequam hoc argumentum relinquamus, aptiorem aperiamus modum Tangentes & Secantes quorumvis Angulorum inveniendi, quam Caput præcedens suppeditat. Quanquam enim Tangentes & Secantes per Sinus & Cosinus determinantur; tamen hoc fit per divisionem, quæ operatio in tantis numeris nimis est operosa. Ac Tangentes quidem & Cotangentes jam supra (§. 136.) exhibuimus, verum illo loco rationem formularum reddere non licuit, quam huic Capiti reservavimus.

198. Ex §. 181. ergo primum expressionem pro Tangente Anguli $\frac{m}{2n}\pi$ elicimus. Cum enim sit $\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\&c. = \frac{\varpi}{4mn}.\,tang.\,\frac{m}{2n}\varpi$ erit $tang.\,\frac{m}{2n}\varpi = \frac{4mn}{\varpi}\left(\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\&c.\right)$. Cum deinde sit $\frac{1}{nn-mm}+\frac{1}{4nn-mm}+\frac{1}{9nn-mm}+\&c. = \frac{1}{2mm}-\frac{\varpi}{2mn}.cot.\,\frac{m}{n}\varpi$, si pro n scribamus $2n$ erit

$$cot.\,\frac{m}{2n}\pi = \frac{2n}{m\varpi}-\frac{4mn}{\varpi}\left(\frac{1}{4nn-mm}+\frac{1}{16nn-mm}+\frac{1}{36nn-mm}\right.$$

LIB. I. $\frac{1}{36nn - mm} + \&c.$). Convertantur hæ fractiones, præter primas, quippe quæ facile in computum ducuntur, in Series infinitas, erit

$$tang. \frac{m}{2n}\pi = \frac{mn}{nn - mm} \cdot \frac{4}{\pi}$$
$$+ \frac{4}{\pi}\left(\frac{m}{3^2 n} + \frac{m^3}{3^4 n^3} + \frac{m^5}{3^6 n^5} + \&c.\right)$$
$$+ \frac{4}{\pi}\left(\frac{m}{5^2 n} + \frac{m^3}{5^4 n^3} + \frac{m^5}{5^6 n^5} + \&c.\right)$$
$$+ \frac{4}{\pi}\left(\frac{m}{7^2 n} + \frac{m^3}{7^4 n^3} + \frac{m^5}{7^6 n^5} + \&c.\right.$$
$$\&c.$$

$$cot. \frac{m}{2n}\varpi = \frac{n}{m} \cdot \frac{2}{\varpi} - \frac{mn}{4nn - mm} \cdot \frac{4}{\varpi}$$
$$- \frac{4}{\pi}\left(\frac{m}{4^2 n} + \frac{m^3}{4^4 n^3} + \frac{m^5}{4^6 n^5} + \&c.\right)$$
$$- \frac{4}{\pi}\left(\frac{m}{6^2 n} + \frac{m^3}{6^4 n^3} + \frac{m^5}{6^6 n^5} + \&c.\right)$$
$$- \frac{4}{\pi}\left(\frac{m}{8^2 n} + \frac{m^3}{8^4 n^3} + \frac{m^5}{8^6 n^5} + \&c.\right)$$
$$\&c.$$

198. At ex valore ipsius ϖ cognito reperitur

$$\frac{1}{\pi} = 0,\ 31830988618379067153776792674502872 4,$$

deinde hic eædem Series occurrunt, quas supra litteris A, B, C, D, &c., & $\alpha, \beta, \gamma, \delta$, &c., indicavimus. His ergo notatis, erit

$$tang. \frac{m}{2n}\pi = \frac{mn}{nn - mm} \cdot \frac{4}{\pi} + \frac{m}{n} \cdot \frac{4}{\pi}(A - 1) + \frac{m^3}{n^3} \times \frac{4}{\pi}(B - 1) + \frac{m^5}{n^5} \cdot \frac{4}{\pi}(C - 1) + \frac{m^7}{n^7} \cdot \frac{4}{\pi}(D - 1)\ \&c.$$

Deinde erit pro Cotangente

cot.

$$cot. \frac{m}{2n}\varpi = \frac{n}{m} \cdot \frac{2}{\varpi} - \frac{4mn}{4mm - nn} \cdot \frac{1}{\varpi} - \frac{m}{n} \cdot \frac{4}{\pi}(\alpha - \frac{1}{2^2}) - \frac{m^3}{n^3} \cdot \frac{4}{\varpi}(\beta - \frac{1}{2^4}) - \frac{m^5}{n^5} \cdot \frac{4}{\pi}(\gamma - \frac{1}{2^6}) - \&c.;$$

atque ex his formulis natæ ſunt expreſſiones, quas ſupra (§. 135.) pro Tangente & Cotangente dedimus; ſimul vero (§. 137.) oſtendimus, quomodo ex Tangentibus & Cotangentibus inventis per ſolam additionem & ſubtractionem Secantes & Coſecantes reperiantur. Harum ergo regularum ope univerſus Canon Sinuum, Tangentium & Secantium, eorumque Logarithmorum multo facilius ſupputari poſſet, quam quidem hoc a primis conditoribus eſt factum.

CAPUT XII.

De reali Functionum fractarum evolutione.

199. Jam ſupra, in Capite ſecundo, methodus eſt tradita Functionem quamcunque fractam in tot partes reſolvendi quot ejus denominator habeat Factores ſimplices; hi enim præbent denominatores fractionum illarum partialium. Ex quo manifeſtum eſt, ſi denominator quos habeat Factores ſimplices imaginarios, fractiones quoque inde ortas fore imaginarias: his ergo caſibus parum juvabit fractionem realem in imaginarias reſolviſſe. Cum igitur oſtendiſſem omnem Functionem integram, qualis eſt denominator cujuſvis fractionis, quantumvis Factoribus ſimplicibus imaginariis ſcateat, tamen in Factores duplices, ſeu ſecundæ dimenſionis, reales ſemper reſolvi poſſe; hoc modo in reſolutione fractionum quantitates imaginariæ evitari poterunt, ſi pro denominatoribus fractionum partialium non Factores denominatoris principalis ſimplices, ſed duplices reales aſſumamus.

LIB. I.

200. Sit igitur propoſita hæc Functio fracta $\frac{M}{N}$, ex qua tot fractiones ſimplices ſecundum methodum ſupra expoſitam eliciantur, quot denominator N habuerit Factores ſimplices reales. Sit autem, loco imaginariorum, hæc expreſſio $pp - 2pqz\,cof.\,\phi + qqzz$ Factor ipſius N; &, quoniam in hoc negotio numeratorem & denominatorem in forma evoluta contemplari oportet, ſit hæc fractio propoſita

$$\frac{A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.}{(pp - 2pqz.\,cof.\,\phi + qqzz)(\alpha + \beta z + \gamma zz + \delta z^3 + \&c.)},$$

ac ponatur fractio partialis ex denominatoris Factore $pp - 2pqz.\,cof.\,\phi + qqzz$ oriunda hæc: $\frac{\mathrm{A} + \mathrm{a}z}{pp - 2pqz.\,cof.\,\phi + qqzz}$, quoniam enim variabilis z in denominatore duas habet dimenſiones, in numeratore unam habere poterit, non vero plures; alias enim integra Functio contineretur, quam ſeorſim elici oportet.

201. Sit, brevitatis gratia, numerator $A + Bz + Cz^2 + \&c. = M$ & alter denominatoris Factor $\alpha + \beta z + \gamma z^2 + \&c. = \mathrm{Z}$; ponatur altera pars ex denominatoris Factore Z oriunda $= \frac{\Upsilon}{\mathrm{Z}}$, eritque $\Upsilon = \frac{M - \mathrm{AZ} - \mathrm{a}Zz}{pp - 2pqz.cof.\,\phi + qqzz}$, quæ expreſſio Functio integra ipſius z eſſe debet, ideoque neceſſe eſt ut $M - \mathrm{AZ} - \mathrm{a}z\mathrm{Z}$ diviſibile ſit per $pp - 2pqz.cof.\,\phi + qqzz$. Evaneſcet ergo $M - \mathrm{AZ} - \mathrm{a}z\mathrm{Z}$, ſi ponatur $pp - 2pqz \times cof.\,\phi + qqzz = 0$, hoc eſt ſi ponatur tam $z = \frac{p}{q}(cof.\,\phi + \surd -1.\,fin.\,\phi)$ quam $z = \frac{p}{q}(cof.\,\phi - \surd -1.\,fin.\,\phi)$; ſit $\frac{p}{q} = f$, eritque $z^n = f^n(cof.\,n\phi \pm \surd -1.\,fin.\,n\phi)$. Duplex ergo hic valor pro z ſubſtitutus duplicem dabit æquationem, unde ambas incognitas conſtantes A & a definire licet.

202. Facta ergo hac ſubſtitutione, æquatio $M = \mathrm{AZ} + \mathrm{a}Zz$ evoluta hanc duplicem dabit æquationem

$A +$

$$\left.\begin{array}{l} A + Bf.\,cos.\,\phi + Cff.\,cos.\,2\,\phi + Df^3.\,cos.\,3\phi + \&c. \\ \pm (Bf.\,sin.\,\phi + Cff.\,sin.\,2\,\phi + Df^3.\,sin.\,3\phi + \&c.)\sqrt{-1} \end{array}\right\} =$$ CAP. XII.

$$\left\{\begin{array}{l} \mathrm{A}(\alpha + \beta f.\,cos.\,\phi + \gamma ff.\,cos.\,2\,\phi + \delta f^3.\,cos.\,3\,\phi + \&c.) \\ \pm\ \mathrm{A}\,(\beta f.\,sin.\,\phi + \gamma ff.\,sin.\,2\,\phi + \delta f^3.\,sin.\,3\,\phi + \&c.)\sqrt{-1} \\ +\ \mathfrak{a}\,(\alpha f.\,cos.\,\phi + \beta ff.\,cos.\,2\phi + \gamma f^3.\,cos.\,3\,\phi + \&c.) \\ \pm\ \mathfrak{a}\,(\alpha f.\,sin.\,\phi + \beta ff.\,sin.\,2\,\phi + \gamma f^3.\,sin.\,3\,\phi + \&c.)\sqrt{-1} \end{array}\right.$$

Sit, ad calculum abbreviandum,

$$\begin{aligned} A + Bf.\,cos.\,\phi + Cff.\,cos.\,2\,\phi + Df^3.\,cos.\,3\,\phi + \&c. &= \mathrm{P} \\ Bf.\,sin.\,\phi + Cff.\,sin.\,2\,\phi + Df^3.\,sin.\,3\,\phi + \&c. &= \mathfrak{p} \\ \alpha + \beta f.\,cos.\,\phi + \gamma ff.\,cos.\,2\,\phi + \delta f^3.\,cos.\,3\,\phi + \&c. &= \mathrm{Q} \\ \beta f.\,sin.\,\phi + \gamma ff.\,sin.\,2\,\phi + \delta f^3.\,sin.\,3\,\phi + \&c. &= \mathfrak{q} \\ \alpha f.\,cos.\,\phi + \beta ff.\,cos.\,2\,\phi + \gamma f^3.\,cos.\,3\,\phi + \&c. &= \mathrm{R} \\ \alpha f.\,sin.\,\phi + \beta ff.\,sin.\,2\,\phi + \gamma f^3.\,sin.\,3\,\phi + \&c. &= \mathfrak{r} \end{aligned}$$

eritque, his positis,

$$\mathrm{P} \pm \mathfrak{p}\sqrt{-1} = \mathrm{AQ} \pm \mathrm{A}\mathfrak{q}\sqrt{-1} + \mathfrak{a}\mathrm{R} \pm \mathfrak{a}\mathfrak{r}\sqrt{-1}.$$

203. Ob signorum ambiguitatem hæ duæ oriuntur æquationes,

$$\begin{aligned} \mathrm{P} &= \mathrm{AQ} + \mathfrak{a}\mathrm{R} \\ \mathfrak{p} &= \mathrm{A}\mathfrak{q} + \mathfrak{a}\mathfrak{r} \end{aligned}$$

ex quibus incognitæ A & $\mathfrak{a}$ ita definiuntur, ut sit

$$\mathrm{A} = \frac{\mathrm{P}\mathfrak{r} - \mathfrak{p}\mathrm{R}}{\mathrm{Q}\mathfrak{r} - \mathfrak{q}\mathrm{R}} \ \&\ \mathfrak{a} = \frac{\mathrm{P}\mathfrak{q} - \mathfrak{p}\mathrm{Q}}{\mathfrak{q}\mathrm{R} - \mathrm{Q}\mathfrak{r}}.$$

Proposita ergo fractione $\frac{M}{(pp - 2pqz.\,cos.\,\phi + qqzz)Z}$ per sequentem regulam fractio partialis ex ea oriunda $\frac{\mathrm{A} + \mathfrak{a}z}{pp + 2pqz.\,cos.\,\phi + qqzz}$ definietur. Posito $f = \frac{p}{q}$, & evolutis singulis terminis, fiat ut sequitur,

 posito

LIB. I. posito $z^n = f^n .\cos. n\phi$, fit $M = P$

$\ldots\ldots z^n = f^n .\sin. n\phi$, fit $M = p$

$\ldots\ldots z^n = f^n .\cos. n\phi$, fit $Z = Q$

$\ldots\ldots z^n = f^n .\sin. n\phi$, fit $Z = q$

$\ldots\ldots z^n = f^n .\cos. n\phi$, fit $zZ = R$

$\ldots\ldots z^n = f^n .\sin. n\phi$, fit $zZ = r$

Inventis hoc modo valoribus P, Q, R, p, q, r erit

$$A = \frac{Pr - pR}{Qr - qR}, \ \& \ a = \frac{pQ - Pq}{Qr - qR}.$$

EXEMPLUM I.

Si fuerit propofita hæc Functio fracta $\frac{zz}{(1 - z + zz)(1 + z^4)}$ ex qua partem a denominatoris Factore $1 - z + zz$ oriundam definire oporteat, quæ fit $\frac{A + az}{1 - z + zz}$. Ac primo quidem hic Factor, cum forma generali $pp - 2pqz.\cos.\phi + qqzz$ comparatus, dat $p = 1$, $q = 1$ & $\cos.\phi = \frac{1}{2}$, unde fit $\phi = 60° = \frac{\pi}{3}$. Quia itaque eft $M = zz$; $Z = 1 + z^4$ & $f = 1$ erit

$$P = \cos.\frac{2}{3}\pi = -\frac{1}{2}; \quad p = \frac{\sqrt{3}}{2}$$

$$Q = 1 + \cos.\frac{4}{3}\pi = \frac{1}{2}; \quad q = -\frac{\sqrt{3}}{2}$$

$$R = \cos.\frac{\pi}{3} + \cos.\frac{5\pi}{3} = 1 \quad ; \quad r = 0.$$

Ex his invenitur $A = -1$; & $a = 0$, ideoque fractio quæfita eft $\frac{-1}{1 - z + zz}$, hujufque complementum erit

1 + z

$\frac{1+z+zz}{1+z^4}$, cujus denominator $1+z^4$ cum habeat Factores $1+z\sqrt{2}+zz$ & $1-z\sqrt{2}+zz$, resolutio denuo suscipi potest; fit autem $\phi = \frac{\pi}{4}$ & priori casu $f = -1$ posteriori $f = +1$. CAP. XII.

EXEMPLUM. II.

Sit igitur proposita hæc fractio resolvenda

$$\frac{1+z+zz}{(1+z\sqrt{2}+zz)(1-z\sqrt{2}+zz)}$$

& erit $M = 1+z+zz$; & pro priore Factore habebitur $f = -1$; $\phi = \frac{\pi}{4}$, & $Z = 1-z\sqrt{2}+zz$, unde erit

$$P = 1 - cos. \frac{\pi}{4} + cos. \frac{2\pi}{4} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\mathfrak{p} = - sin. \frac{\pi}{4} + sin. \frac{2\pi}{4} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$Q = 1 + \sqrt{2}.cos. \frac{\pi}{4} + cos. \frac{2\pi}{4} = 2$$

$$\mathfrak{q} = +\sqrt{2}.sin. \frac{\pi}{4} + sin. \frac{2\pi}{4} = 2$$

$$R = -cos. \frac{\pi}{4} - \sqrt{2}.cos. \frac{2\pi}{4} - cos. \frac{3\pi}{4} = 0$$

$$\mathfrak{r} = -sin. \frac{\pi}{4} - \sqrt{2}.sin. \frac{2\pi}{4} - sin. \frac{3\pi}{4} = -2\sqrt{2}$$

Ex his reperitur $Q\mathfrak{r} - \mathfrak{q}R = -4\sqrt{2}$: &

$A = \frac{\sqrt{2}-1}{2\sqrt{2}}$, & $\mathfrak{a} = 0$ unde ex denominatoris Factore $1+z\sqrt{2}+zz$ hæc orietur fractio partialis $\frac{(\sqrt{2}-1):2\sqrt{2}}{1+z\sqrt{2}+zz}$, alter autem Factor dabit simili modo hanc $\frac{(\sqrt{2}+1):2\sqrt{2}}{1-z\sqrt{2}+zz}$.

Hinc Functio primum proposita $\frac{zz}{(1-z+zz)(1+z^4)}$ resolvitur

 vitur in has $\frac{-1}{1-z+zz} + \frac{(\surd 2 - 1):2\surd 2}{1+z\surd 2+zz} + \frac{(\surd 2+1):2\surd 2}{1-z\surd 2+zz}$

EXEMPLUM III.

Sit propoſita hæc fractio reſolvenda

$$\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}$$

Pro Factore denominatoris $1-\frac{8}{5}z+zz$ oriatur iſta fractio $\frac{A+\mathfrak{A}z}{1-\frac{8}{5}z+zz}$; eritque $p=1$; $q=1$; $cof.\,\Phi=\frac{4}{5}$, unde $f=1$; $M=1+2z+zz$; $Z=1+2z+3zz$. Quia vero hic ratio Anguli Φ ad rectum non conſtat, Sinus & Coſinus ejus multiplorum ſeorſim debent inveſtigari. Cum ſit

$$cof.\,\Phi=\frac{4}{5};\ \text{erit}\ fin.\,\Phi=\frac{3}{5}$$

$$cof.\,2\Phi=\frac{7}{25};\qquad fin.\,2\Phi=\frac{24}{25}$$

$$cof.\,3\Phi=\frac{-44}{125};\qquad fin.\,3\Phi=\frac{117}{125};$$

hinc fit

$$P=1+2.\frac{4}{5}+\frac{7}{25}=\frac{72}{25}$$

$$\mathfrak{P}=2.\frac{3}{5}+\frac{24}{25}=\frac{54}{25}$$

$$Q=1+2.\frac{4}{5}+3.\frac{7}{25}=\frac{86}{25}$$

$$\mathfrak{Q}=2.\frac{3}{5}+3.\frac{24}{25}=\frac{102}{25}$$

$$R=\frac{4}{5}+2.\frac{7}{25}-3.\frac{44}{125}=\frac{38}{125}$$

$$\mathfrak{R}=\frac{3}{5}+2.\frac{24}{25}+3.\frac{117}{125}=\frac{666}{125}$$

ideoque $Q\mathfrak{R}-\mathfrak{Q}R=\frac{53400}{25.125}=\frac{2136}{125}$. Ergo

A =

$$A = \frac{1836}{2136} = \frac{153}{178};\quad \mathfrak{A} = -\frac{540}{2136} = -\frac{45}{178}.$$ CAP. XII.

Quare fractio ex Factore $1 - \frac{8}{5}z + zz$ oriunda erit

$$\frac{9(17 - 5z) : 178}{1 - \frac{8}{5}z + zz}.$$

Quæramus simili modo fractionem alteri Factori respondentem; erit $p = 1$, $q = -\sqrt{3}$ & $cos.\, \phi = \frac{1}{\sqrt{3}}$, ergo $f = -\frac{1}{\sqrt{3}}$, $M = 1 + 2z + zz$ & $Z = 1 - \frac{8}{5}z + zz$. Fiet autem, ob $cos.\, \phi = \frac{1}{\sqrt{3}}$, $fin.\, \phi = \frac{\sqrt{2}}{\sqrt{3}}$

$$cos.\, 2\phi = -\frac{1}{3},\quad fin.\, 2\phi = \frac{2\sqrt{2}}{3}$$

$$cos.\, 3\phi = -\frac{5}{3\sqrt{3}},\quad fin.\, 3\phi = \frac{\sqrt{3}}{3\sqrt{3}}$$

consequenter

$$P = 1 - \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{3} \cdot -\frac{1}{3} = \frac{2}{9}$$

$$\mathfrak{P} = -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3} \cdot \frac{2\sqrt{2}}{3} = -\frac{4\sqrt{2}}{9}$$

$$Q = 1 + \frac{8}{5\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{3} \cdot -\frac{1}{3} = \frac{64}{45}$$

$$\mathfrak{Q} = +\frac{8}{5\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3} \cdot \frac{2\sqrt{2}}{3} = \frac{34\sqrt{2}}{45}$$

$$R = -\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} - \frac{8}{5 \cdot 3} \cdot -\frac{1}{3} - \frac{1}{3\sqrt{3}} \cdot -\frac{5}{3\sqrt{3}} = \frac{4}{135}$$

$$\mathfrak{R} = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} - \frac{8}{5 \cdot 3} \cdot \frac{2\sqrt{2}}{3} - \frac{1}{3\sqrt{3}} \cdot \frac{\sqrt{2}}{3\sqrt{3}} = -\frac{98\sqrt{2}}{135}$$

ideoque $Q\mathfrak{R} - \mathfrak{Q}R = -\frac{712\sqrt{2}}{675}$; fiet ergo

$$A = \frac{100}{712} = \frac{25}{178};\quad \mathfrak{A} = \frac{540}{712} = \frac{135}{178}.$$

Fractio

LIB. I.

Fractio ergo proposita $\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}$ resolvitur in $\frac{9(17-5z):178}{1-\frac{8}{5}z+zz}+\frac{5(5+27z):178}{1+2z+3zz}$.

204. Possunt autem valores litterarum R & ʀ ex litteris Q & ǫ definiri, cum enim sit

$$Q = \alpha + \beta f.\,cos.\,\Phi + \gamma f^2.\,cos.\,2\Phi + \delta f^3.\,cos.\,3\Phi \text{ \&c.}$$
$$\mathfrak{q} = \beta f.\,sin.\,\Phi + \gamma f^2.\,sin.\,2\Phi + \delta f^3.\,sin.\,3\Phi \text{ \&c.}$$

erit

$$Q.\,cos.\Phi - \mathfrak{q}.\,sin.\Phi = \alpha.\,cos.\Phi + \beta f.\,cos.2\Phi + \gamma f^2.\,cos.3\Phi + \text{\&c.}$$

ideoque $R = f(Q.\,cos.\Phi - \mathfrak{q}.\,sin.\,\Phi)$

deinde erit

$$Q.\,sin.\Phi + \mathfrak{q}.\,cos.\Phi = \alpha.\,sin.\,\Phi + \beta f.\,sin.2\Phi + \gamma ff.\,sin.\,3\Phi + \text{\&c.}$$

ergo $\mathfrak{r} = f(Q.\,sin.\,\Phi + \mathfrak{q}.\,cos.\,\Phi)$

Ex his porro fit

$$Q\mathfrak{r} - \mathfrak{q}R = (QQ + \mathfrak{qq})f.\,sin.\,\Phi$$
$$P\mathfrak{r} - \mathfrak{p}R = (PQ + \mathfrak{pq})f.sin.\,\Phi + (P\mathfrak{q} - \mathfrak{p}Q)f.cos.\,\Phi$$

eritque consequenter

$$A = \frac{PQ + \mathfrak{pq}}{QQ + \mathfrak{qq}} + \frac{P\mathfrak{q} - \mathfrak{p}Q}{QQ + \mathfrak{qq}}.\frac{cos.\,\Phi}{sin.\,\Phi}$$
$$\mathfrak{a} = -\frac{P\mathfrak{q} + \mathfrak{p}Q}{(QQ + \mathfrak{qq})f.sin.\,\Phi}.$$

Quare ex denominatoris Factore $pp - 2pqz.cos.\,\Phi + qqzz$ nascitur ista fractio partialis

$$\frac{(PQ + \mathfrak{pq})\,f.\,sin.\,\Phi + (P\mathfrak{q} - \mathfrak{p}Q)\,(f.\,cos.\,\Phi - z)}{(pp - 2pqz.\,cos.\,\Phi + qqzz)\,(QQ + \mathfrak{qq})\,f.\,sin.\,\Phi}$$

seu, ob $f = \frac{p}{q}$, hæc

$$\frac{(PQ + \mathfrak{pq})\,p.\,sin.\,\Phi + (P\mathfrak{q} - \mathfrak{p}Q)\,(p.cos.\,\Phi - qz)}{(pp - 2pqz.\,cos.\,\Phi + qqzz)\,(QQ + \mathfrak{qq})\,p.\,sin.\,\Phi}.$$

205. Oritur ergo hæc fractio partialis ex Functionis propositæ $\frac{M}{(pp - 2pqz.\,cos.\,\Phi + qqzz)Z}$ Factore denominatoris $pp - 2pqz.\,cos.\,\Phi + qqzz$, atque litteræ P, ᴘ, Q & ǫ sequenti modo ex Functionibus M & Z inveniuntur:

posito

posito $z^n = \frac{p^n}{q^n} . \cos. n\Phi$, fit $M = \mathrm{P}$,

& $Z = \mathrm{Q}$;

& posito $z^n = \frac{p^n}{q^n} . \sin. n\Phi$, fit $M = \mathrm{p}$,

& $Z = \mathrm{q}$:

ubi notandum est Functiones M & Z, antequam hæc substitutio fiat, omnino evolvi debere, ut hujusmodi habeant formas

$$M = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.,$$
$$\& \; Z = \alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \&c.;$$

eritque ideo

$$\mathrm{P} = A + B\frac{p}{q}.\cos.\Phi + C\frac{p^2}{q^2}.\cos.2\Phi + D\frac{p^3}{q^3}.\cos.3\Phi + \&c.$$
$$\mathrm{p} = \quad B\frac{p}{q}.\sin.\Phi + C\frac{p^2}{q^2}.\sin.2\Phi + D\frac{p^3}{q^3}.\sin.3\Phi + \&c.$$
$$\mathrm{Q} = \alpha + \beta\frac{p}{q}.\cos.\Phi + \gamma\frac{p^2}{q^2}.\cos.2\Phi + \delta\frac{p^3}{q^3}.\cos.3\Phi + \&c.$$
$$\mathrm{q} = \quad \beta\frac{p}{q}\sin.\Phi + \gamma\frac{p^2}{q^2}.\sin.2\Phi + \delta\frac{p^3}{q^3}.\sin.3\Phi + \&c..$$

206. Ex præcedentibus autem intelligitur hanc resolutionem locum habere non posse, si Functio Z eundem Factorem $pp - 2pqz.\cos.\Phi + qqzz$ adhuc in se complectatur; hoc enim casu in æquatione $M = \mathrm{A}Z + \mathrm{a}Zz$ facta substitutione $z^n = f^n(\cos.n\Phi \pm \surd -1.\sin.\Phi)$, ipsa quantitas Z evanesceret, nihilque propterea colligi posset. Quamobrem, si Functionis fractæ $\frac{M}{N}$ denominator habeat Factorem $(pp - 2pqz.\cos.\Phi + qqzz)^2$ vel altiorem Potestatem, peculiari opus erit resolutione. Sit igitur $N = (pp - 2pqz.\cos.\Phi + qqzz)^2 Z$; atque ex denominatoris Factore $(pp - 2pqz.\cos.\Phi + qqzz)^2$ orientur hujusmodi duæ fractiones partiales

Lib. I. $\frac{\mathrm{A}+\mathfrak{a}z}{(pp-2pqz.\cos.\phi+qqzz)^2}+\frac{\mathrm{B}+\mathfrak{b}z}{pp-2pqz.\cos.\phi+qqzz}$,

ubi litteras constantes A, $\mathfrak{a}$, B, $\mathfrak{b}$ determinari oportet.

207. His positis, debebit ista expressio

$$\frac{M-(\mathrm{A}+\mathfrak{a}z)Z-(\mathrm{B}+\mathfrak{b}z)Z(pp-2pqz.\cos.\phi+qqzz)}{(pp-2pqz.\cos.\phi+qqzz)^2}$$

esse Functio integra, & hanc ob rem numerator divisibilis erit per denominatorem. Primum ergo hæc expressio $M-\mathrm{A}Z-\mathfrak{a}zZ$ divisibilis esse debet per $pp-2pqz.\cos.\phi+qqzz$; qui cum sit casus præcedens, eodem quoque modo litteræ A & $\mathfrak{a}$ determinabuntur.

Quare, posito $z^n=\frac{p^n}{q^n}.\cos.n\phi$, sit $M=\mathrm{P}$,

& $Z=\mathrm{N}$:

&, posito $z^n=\frac{p^n}{q^n}.\sin.n\phi$, sit $M=\mathfrak{p}$,

& $Z=\mathfrak{n}$.

Hisque factis secundum regulam supra datam, erit

$$\mathrm{A}=\frac{\mathrm{P}\mathrm{N}+\mathfrak{p}\mathfrak{n}}{\mathrm{N}^2+\mathfrak{n}^2}+\frac{\mathrm{P}\mathfrak{n}-\mathfrak{p}\mathrm{N}}{\mathrm{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\phi}{\sin.\phi}$$

$$\mathfrak{a}=-\frac{\mathrm{P}\mathfrak{n}+\mathfrak{p}\mathrm{N}}{\mathrm{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\phi}.$$

208. Inventis ergo hoc modo A & $\mathfrak{a}$, fiet $\frac{M-(\mathrm{A}+\mathfrak{a}z)Z}{pp-2pqz.\cos.\phi+qqzz}$ Functio integra, quæ sit $=P$; atque superest ut $P-\mathrm{B}Z-\mathfrak{b}zZ$ divisibile evadat per $pp-2pqz.\cos.\phi+qqzz$, quæ expressio cum similis sit præcedenti, si

posito $z^n=\frac{p^n}{q^n}.\cos.n\phi$, vocetur $P=\mathrm{R}$,

&, posito $z^n=\frac{p^n}{q^n}.\sin.n\phi$, vocetur $P=\mathfrak{r}$; erit

B =

$$B = \frac{RN + rN}{N^2 + N^2} + \frac{RN - rN}{N^2 + N^2} \cdot \frac{cos.\Phi}{sin.\Phi}$$

$$b = -\frac{RN + rN}{N^2 + N^2} \cdot \frac{q}{p\,sin.\Phi}.$$

209. Hinc jam generaliter concludere licet quomodo resolutio institui debeat, si denominator Functionis propositæ $\frac{M}{N}$, Factorem habeat $(pp - 2pqz.\,cos.\,\Phi + qqzz)^k$: sit enim $N = (pp - 2pqz.\,cos.\Phi + qqzz)^k Z$, ita ut hæc resolvenda sit Functio fracta

$$\frac{M}{(pp - 2pqz.cos.\Phi + qqzz)^k Z}.$$

Præbeat ergo Factor denominatoris $(pp - 2pqz.\,cos.\Phi + qqzz)^k$ has partes:

$$\frac{A + az}{(pp - 2pqz.\,cos.\,\Phi + qqzz)^k} + \frac{B + bz}{(pp - 2pqz.\,cos.\,\Phi + qqzz)^{k-1}} +$$

$$\frac{C + cz}{(pp - 2pqz.cos.\Phi + qqzz)^{k-2}} + \frac{D + dz}{(pp - 2pqz.cos.\Phi + qqzz)^{k-3}} + \&c.$$

Jam, posito $z^n = \frac{p^n}{q^n}.\,cos.n\Phi$, sit $M = \mathrm{M}$,

& $Z = \mathrm{N}$;

&, posito $z^n = \frac{p^n}{q^n}.\,sin.\,n\Phi$, sit $M = \mathrm{m}$;

& $Z = \mathrm{n}$;

erit

$$A = \frac{MN + mn}{N^2 + n^2} + \frac{Mn - mN}{N^2 + n^2} \cdot \frac{cos.\,\Phi}{sin.\,\Phi}$$

$$a = -\frac{Mn + mN}{N^2 + n^2} \cdot \frac{q}{p\,sin.\Phi}.$$

Deinde vocetur $\dfrac{M - (A + az)Z}{pp - 2ppz.\,cos.\Phi + qqzz} = P$; atque,

Y z posito

LIB. I.

posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\Phi$, fit $P = \mathrm{P}$,

&, posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\Phi$, fit $P = \mathrm{p}$;

erit

$$\mathrm{B} = \frac{\mathrm{PN} + \mathrm{pn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Pn} - \mathrm{pN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$\mathrm{b} = - \frac{\mathrm{Pn} + \mathrm{pN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p \sin. \Phi}.$$

Tum vocetur $\frac{P - (\mathrm{B} + \mathrm{b}z) Z}{pp - 2pqz. \cos. \Phi + qqzz} = Q$, atque

posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\Phi$, fit $Q = \mathrm{Q}$,

&, posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\Phi$, fit $Q = \mathrm{q}$;

erit

$$\mathrm{C} = \frac{\mathrm{QN} + \mathrm{qn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Qn} - \mathrm{qN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$\mathrm{c} = - \frac{\mathrm{Qn} + \mathrm{qN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p \sin. \Phi}.$$

Porro vocetur $\frac{Q - (\mathrm{C} + \mathrm{c}z) Z}{pp - 2pqz. \cos. \Phi + qqzz} = R$, atque

posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\Phi$, fit $R = \mathrm{R}$,

&, posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\Phi$, fit $R = \mathrm{r}$;

erit

$$\mathrm{D} = \frac{\mathrm{RN} + \mathrm{rn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Rn} - \mathrm{rN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$\mathrm{d} = - \frac{\mathrm{Rn} + \mathrm{rN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p \sin. \Phi}.$$

hocque

hocque modo progrediendum est donec ultimæ fractionis, cujus denominator est $pp - 2pqz.\,cos.\,\Phi + qqzz$, numerator fuerit determinatus. CAP. XII.

EXEMPLUM.

Sit ista proposita Functio fracta

$$\frac{z - z^3}{(1+zz)^4(1+z^4)}$$

ex cujus denominatoris Factore $(1+zz)^4$ oriantur hæ fractiones partiales,

$$\frac{\mathrm{A} + \text{\textsc{a}}z}{(1+zz)^4} + \frac{\mathrm{B} + \text{\textsc{b}}z}{(1+zz)^3} + \frac{\mathrm{C} + \text{\textsc{c}}z}{(1+zz)^2} + \frac{\mathrm{D} + \text{\textsc{d}}z}{1+zz}.$$

Comparatione ergo instituta, erit $p = 1$, $q = 1$, $cos.\Phi = 0$; ideoque $\Phi = \frac{1}{2}\pi$, porroque $M = z - z^3$ & $Z = 1 + z^4$. Hinc erit $\mathrm{M} = 0$; $\text{\textsc{m}} = 2$; $\mathrm{N} = 2$; $\text{\textsc{n}} = 0$, & $sin.\,\Phi = 1$.

Hinc itaque invenitur

$$\mathrm{A} = -\frac{4}{4}.\,0 = 0, \ \& \ \text{\textsc{a}} = 1.$$

ergo $\mathrm{A} + \text{\textsc{a}}z = z$; hincque $P = \frac{z - z^3 - z - z^5}{1+zz} = -z^3$,

& $\mathrm{P} = 0$, $\text{\textsc{p}} = 1$: unde reperitur

$$\mathrm{B} = 0, \ \& \ \text{\textsc{b}} = \frac{1}{2}.$$

Ergo $\mathrm{B} + \text{\textsc{b}}z = \frac{1}{2}z$, & $Q = \frac{-z^3 - \frac{1}{2}z - \frac{1}{2}z^5}{1+zz} = -$

$$\frac{1}{2}z - \frac{1}{2}z^3;$$

unde $\mathrm{Q} = 0$ & $\text{\textsc{q}} = 0$, ergo

$\mathrm{C} = 0$ & $\text{\textsc{c}} = 0$, hincque $R = -$

$$\frac{\frac{1}{2}z - \frac{1}{2}z^3}{1+zz} = -\frac{1}{2}z,$$

ergo $\mathrm{R} = 0$; $\text{\textsc{r}} = -\frac{1}{2}$; unde fit

$$\mathrm{D} = 0 \ \& \ \text{\textsc{d}} = -\frac{1}{4}.$$

LIB. I. Quamobrem fractiones quæsitæ sunt hæ $\frac{z}{(1+zz)^4}+\frac{z}{2(1+zz)^3}-\frac{z}{4(1+zz)}$. Reliquæ vero fractionis numerator est $= S = \frac{R-(D+\mathfrak{D}z)Z}{1+zz} = -\frac{1}{4}z + \frac{1}{4}z^3$, quæ ergo erit $= \frac{-z+z^3}{4(1+z^4)}$.

210. Hac ergo methodo simul innotescit fractio complementi, quæ cum inventis conjuncta producat fractionem propositam ipsam. Scilicet si fractionis

$$\frac{M}{(pp-2pqz.\cos.\Phi+qqzz)^k Z}$$

inventæ fuerint omnes fractiones partiales ex Factore $(pp - 2pqz.\cos.\Phi + qqzz)^k$ oriundæ, pro quibus formati sunt valores Functionum P, Q, R, S, T, si harum litterarum Series ulterius continuetur, erit ea, quæ ultimam, qua opus est ad numeratores inveniendos, sequitur, numerator reliquæ fractionis denominatorem Z habentis; nempe, si $k=1$, erit reliqua fractio $\frac{P}{Z}$; si $k=2$, erit reliqua fractio $\frac{Q}{Z}$; si $k=3$, erit ea $\frac{R}{Z}$, & ita porro. Inventa autem hac reliqua fractione denominatorem Z habente, ea per has regulas ulterius resolvi poterit.

CAPUT

CAPUT XIII.

De Seriebus recurrentibus.

211. AD hoc Serierum genus, quas MOIVRÆUS *recurrentes* vocare ſolet, hic refero omnes Series quæ ex evolutione Functionis cujuſque fractæ per diviſionem actualem inſtituta naſcuntur. Supra enim jam oſtendimus has Series ita eſſe comparatas, ut quivis terminus ex aliquot præcedentibus ſecundum legem quandam conſtantem determinetur, quæ lex a denominatore Functionis fractæ pendet. Cum autem nunc Functionem quamcunque fractam in alias ſimpliciores reſolvere docuerim, hinc Series quoque recurrens in alias ſimpliciores reſolvetur. In hoc igitur Capite propoſitum eſt Serierum recurrentium cujuſvis gradus reſolutionem in ſimpliciores exponere.

212. Sit propoſita iſta Functio fracta genuina

$$\frac{a + bz + czz + dz^3 + \&c.}{1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 - \&c.}$$

quæ per diviſionem evolvatur in hanc Seriem recurrentem

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes quemadmodum progrediantur, ſupra eſt oſtenſum. Quod ſi jam Functio illa fracta reſolvatur in fractiones ſuas ſimplices, & unaquæque in Seriem recurrentem evolvatur, manifeſtum eſt ſummam omnium harum Serierum ex fractionibus partialibus ortarum æqualem eſſe debere Seriei recurrenti.

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.$$

Fractiones ergo partiales, quas ſupra invenire docuimus, dabunt

LIB. I. bunt Series partiales, quarum indoles ob ſimplicitatem facile perſpicitur; omnes autem Series partiales junctim ſumtæ producent Seriem recurrentem propoſitam; unde & hujus natura penitius cognoſcetur.

213. Sint Series recurrentes ex ſingulis fractionibus partialibus ortæ hæ.

$$\begin{array}{l} a + bz + czz + dz^3 + ez^4 + \&c. \\ a' + b'z + c'zz + d'z^3 + e'z^4 + \&c. \\ a'' + b''z + c''zz + d''z^3 + e''z^4 + \&c. \\ a''' + b'''z + c'''zz + d'''z^3 + e'''z^4 + \&c. \\ \&c. \end{array}$$

Quoniam hæ Series junctim ſumtæ æquales eſſe debent huic

$$A + Bz + Czz + Dz^3 + Ez^4 + \&c.,$$

neceſſe eſt ut ſit

$$\begin{array}{l} A = a + a' + a'' + a''' + \&c. \\ B = b + b' + b'' + b''' + \&c. \\ C = c + c' + c'' + c''' + \&c. \\ D = d + d' + d'' + d''' + \&c. \\ \&c. \end{array}$$

Hinc, ſi ſingularum Serierum ex fractionibus partialibus ortarum definiri queant coëfficientes Poteſtatis z^n, horum ſumma dabit coëfficientem Poteſtatis z^n in Serie recurrente $A + Bz + Cz^2 + D\ ^3 + \&c.$

214. Dubium hic ſuboriri poſſet, an, ſi duæ hujuſmodi Series fuerint inter ſe æquales

$$A + Bz + Cz^2 + Dz^3 + \&c. = \mathrm{A} + \mathrm{B}z + \mathrm{C}z^2 + \mathrm{D}z^3 + \&c.,$$

neceſſario inde ſequatur, coëfficientes ſimilium Poteſtatum ipſius z inter ſe eſſe æquales; ſeu an ſit $A = \mathrm{A}$; $B = \mathrm{B}$; $C = \mathrm{C}$;

$\mathfrak{C} = C$; $\mathfrak{D} = D$; &c.. Hoc autem dubium facile tolletur, si perpendamus hanc æqualitatem subsistere debere quemcunque valorem obtineat variabilis z. Sit igitur $z = 0$, atque manifestum est fore $\mathfrak{A} = A$. His ergo terminis æqualibus utrinque sublatis, ac reliqua æquatione per z divisa, habebitur CAP. XIII.

$$\mathfrak{B} + \mathfrak{C}z + \mathfrak{D}z^2 + \&c. = B + Cz + Dz^2 + \&c.;$$

unde sequitur fore $\mathfrak{B} = B$: simili autem modo ostendetur esse $\mathfrak{C} = C$; $\mathfrak{D} = D$, & ita porro in infinitum.

215. Contemplemur ergo Series, quæ ex fractionibus partialibus, in quas fractio quæpiam proposita resolvitur, oriuntur. Ac primo quidem patet fractionem $\frac{A}{1 - pz}$ dare Seriem $A + Apz + Ap^2z^2 + Ap^3z^3 + \&c.$, cujus terminus generalis est Ap^nz^n; hæc enim expressio vocari solet *terminus generalis*, quoniam ex ea, loco n numeros omnes successive substituendo, omnes Seriei termini nascuntur. Deinde ex fractione $\frac{A}{(1 - pz)^2}$, oritur Series $A + 2Apz + 3Ap^2z^2 + 4Ap^3z^3 + \&c.$, cujus terminus generalis est $(n + 1)Ap^nz^n$. Tum ex fractione $\frac{A}{(1 - pz)^3}$, oritur Series $A + 3Apz + 6Ap^2z^2 + 10Ap^3z^3 + \&c.$, cujus terminus generalis est $\frac{(n + 1)(n + 2)}{1. \quad 2} Ap^nz^n$. Generatim autem fractio $\frac{A}{(1 - pz)^k}$ præbet Seriem hanc $A + kApz + \frac{k(k + 1)}{1. \quad 2.} Ap^2z^2 + \frac{k(k + 1)(k + 2)}{1. \quad 2. \quad 3} Ap^3z^3 + \&c.$, cujus terminus generalis est $\frac{(n + 1)(n + 2)(n + 3) \ldots\ldots (n + k - 1)}{1. \quad 2. \quad 3 \ldots\ldots\ldots (k - 1)} Ap^nz^n$. Ex ipsa autem Seriei progressione colligitur hic idem terminus $= \frac{k(k + 1)(k + 2) \ldots\ldots\ldots (k + n - 1)}{1. \quad 2. \quad 3 \ldots\ldots\ldots n} Ap^nz^n$: hæc vero

LIB. I. expressio illi est æqualis, id quod multiplicatione per crucem instituta patebit, fiet enim,

$$1.2.3\ldots\ldots n(n+1)\ldots\ldots(n+k-1) = 1.2.3\ldots\ldots(k-1)k\ldots\ldots(k+n-1)$$

quæ est æquatio identica.

216. Quoties ergo in resolutione Functionum fractarum ad hujusmodi fractiones partiales $\frac{A}{(1-pz)^k}$ pervenitur, toties Seriei recurrentis ex illa Functione fracta ortæ $A + Bz + Cz^2 + Dz^3 +$ &c., terminus generalis assignari poterit, quippe qui erit summa terminorum generalium Serierum, quæ ex fractionibus partialibus nascuntur.

EXEMPLUM I.

Invenire terminum generalem Seriei recurrentis, quæ ex hac fractione $\frac{1-z}{1-z-2zz}$ *nascitur.*

Series hinc nata est $1 + 0z + 2zz + 2z^3 + 6z^4 + 10z^5 + 22z^6 + 42z^7 + 86z^8 +$ &c.. Ad coëfficientem potestatis generalis z^n inveniendum, fractio $\frac{1-z}{1-z-2zz}$ resolvatur in $\frac{\frac{2}{3}}{1+z} + \frac{\frac{1}{3}}{1-2z}$, unde oritur terminus generalis quæsitus

$$\left(\frac{2}{3}(-1)^n + \frac{1}{3}.2^n\right)z^n = \frac{2^n \pm 2}{3} z^n,$$ ubi signum $+$ valet si n sit numerus par, signum $-$ si n sit impar.

EXEMPLUM II.

Invenire terminum generalem Seriei recurrentis quæ oritur ex fractione $\frac{1-z}{1-5z+6zz}$, *seu Seriei hujus* $1 + 4z + 14z^2 + 46z^3 + 146z^4 + 454z^5 +$ &c.

Ob

Ob denominatorem $= (1 - 2z)(1 - 3z)$ resolvitur fractio in has $\frac{-1}{1-2z} + \frac{2}{1-3z}$, ex quibus fit terminus generalis $2 \cdot 3^n z^n - 2^n z^n = (2 \cdot 3^n - 2^n) z^n$. CAP. XIII.

EXEMPLUM III.

Invenire terminum generalem Seriei hujus $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + 29z^6 + 47z^7 +$ &c., *quæ oritur ex evolutione fractionis* $\frac{1+2z}{1-z-zz}$.

Ob denominatoris Factores $1 - (\frac{1+\sqrt{5}}{2})z$ & $1 - (\frac{1-\sqrt{5}}{2})z$, per resolutionem prodeunt $\frac{\frac{\sqrt{5}+1}{2}}{1-(\frac{1+\sqrt{5}}{2})z} + \frac{\frac{1-\sqrt{5}}{2}}{1-(\frac{1-\sqrt{5}}{2})z}$, unde erit terminus generalis $=$
$(\frac{1+\sqrt{5}}{2})^{n+1} z^n + (\frac{1-\sqrt{5}}{2})^{n+1} z^n$.

EXEMPLUM IV.

Invenire terminum generalem Seriei hujus

$a + (\alpha a + b)z + (\alpha^2 a + \alpha b + \beta a)z^2 + (\alpha^3 a + \alpha^2 b + 2\alpha\beta a + \beta b)z^3 +$ &c,

quæ oritur ex evolutione fractionis $\frac{a+bz}{1-\alpha z-\beta zz}$.

Per resolutionem oriuntur hæ duæ fractiones:

Z 2 (a

LIB. I.

$$\frac{(a(\alpha+\sqrt{(\alpha\alpha+4\beta)})+2b):2\sqrt{(\alpha\alpha+4\beta)}}{1-(\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2})z}+$$

$$\frac{(a(\sqrt{(\alpha\alpha+4\beta)}-\alpha)-2b):2\sqrt{(\alpha\alpha+4\beta)}}{1-(\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2})z},$$ hinc

terminus generalis erit $\frac{a(\sqrt{(\alpha\alpha+4\beta)}+\alpha)+2b}{2\sqrt{(\alpha\alpha+4\beta)}}$ $(\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2})^n z^n+\frac{a(\sqrt{(\alpha\alpha+4\beta)}-\alpha)-2b}{2\sqrt{(\alpha\alpha+4\beta)}}$ $(\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2})^n z^n$; ex quo omnium Serierum recurrentium, quarum quisque terminus per duos præcedentes determinatur, termini generales expedite definiri poterunt.

EXEMPLUM V.

Invenire terminum generalem hujus Seriei $1+z+2z^2+2z^3+3z^4+3z^5+4z^6+4z^7+$ &c., *quæ oritur ex fractione* $\frac{1}{1-z-zz+z^3}=\frac{1}{(1-z)^2(1+z)}$.

Quanquam lex progressionis primo intuitu ita est manifesta ut explicatione non indigeat, tamen fractiones per resolutionem ortæ $\frac{\frac{1}{2}}{(1-z)^2}+\frac{\frac{1}{4}}{1-z}+\frac{\frac{1}{4}}{1+z}$ dant hunc terminum generalem $\frac{1}{2}(n+1)z^n+\frac{1}{4}z^n+\frac{1}{4}(-1)^n z^n=\frac{2n+3\pm1}{4}z^n$, ubi signum superius valet si n fuerit numerus par, inferius si n fuerit impar.

217. Hoc pacto omnium Serierum recurrentium termini generales exhiberi possunt, quoniam omnes fractiones in hujusmodi fractiones partiales simplices resolvere licet. Quod si autem expressiones imaginarias vitare velimus, sæpenumero ad hujusmodi fractiones partiales pervenietur

A +

$\frac{A + Bpz}{1 - 2pz.\,cos.\,\Phi + ppzz}$; $\frac{A + Bpz}{(1 - 2pz.\,cos.\,\Phi + ppzz)^2}$; & $\frac{A + Bpz}{(1 - 2pz.\,cos.\,\Phi + ppzz)^k}$, ex quarum evolutione cujusmodi Series nascantur videndum est. Ac primo quidem, ob $cos.\,n\Phi = 2\,cos.\,\Phi.\,cos.\,(n-1)\Phi - cos.\,(n-2)\Phi$, fractio $\frac{A}{1 - 2pz.\,cos.\,\Phi + ppzz}$ evoluta dabit

$$\begin{aligned} A + 2Apz.\,cos.\,\Phi &+ 2Appzz.\,cos.\,2\Phi + 2Ap^3z^3.\,cos.\,3\Phi + 2Ap^4z^4.\,cos.\,4\Phi \\ &+ Appzz., \quad + 2Ap^3z^3.\,cos.\,\Phi + 2Ap^4z^4.\,cos.\,2\Phi \\ &\qquad\qquad + Ap^4z^4. \quad \&c. \end{aligned}$$

cujus Seriei terminus generalis non tam facile apparet.

218. Quo igitur ad scopum perveniamus, consideremus has duas Series

$$Ppz.\,sin.\,\Phi + Pp^2z^2.\,sin.\,2\Phi + Pp^3z^3.\,sin.\,3\Phi + Pp^4z^4.\,sin.\,4\Phi + \&c.$$
$$Q + Qpz.\,cos.\,\Phi + Qp^2z^2.\,cos.\,2\Phi + Qp^3z^3.\,cos.\,3\Phi + Qp^4z^4.\,cos.\,4\Phi + \&c.$$

quæ duæ Series utique nascuntur ex evolutione fractionis, cujus denominator est $1 - 2pz.\,cos.\,\Phi + ppzz$. Ac prior quidem oritur ex hac fractione $\frac{Ppz.\,sin.\,\Phi}{1 - 2pz.\,cos.\,\Phi + ppzz}$, posterior vero ex hac $\frac{Q - Qpz.\,cos.\,\Phi}{1 - 2pz.\,cos.\,\Phi + ppzz}$. Addantur hæ duæ fractiones, atque summa $\frac{Q + Ppz.\,sin.\,\Phi - Qpz.\,cos.\,\Phi}{1 - 2pz.\,cos.\,\Phi + ppzz}$ dabit Seriem cujus terminus generalis erit $= (P\,sin.\,n\Phi + Q\,cos.\,n\Phi)\,p^n z^n$. Fiat autem hæc fractio propositæ $\frac{A + Bpz}{1 - 2pz.\,cos.\,\Phi + ppzz}$ æqualis, erit $Q = A$, & $P = A\,cot.\,\Phi + B\,cosec.\,\Phi$. Seriei ergo ex hac fractione $\frac{A + Bpz}{1 - 2pz.\,cos.\,\Phi + ppzz}$ ortæ terminus generalis erit $= \frac{A\,cos.\,\Phi\,sin.\,n\Phi + B\,sin.\,n\Phi + A\,sin.\,\Phi.\,cos.\,n\Phi}{sin.\,\Phi}\,p^n z^n = \frac{A\,sin.\,(n+1)\Phi + B\,sin.\,n\Phi}{sin.\,\Phi}\,p^n z^n$.

LIB. I. 219. Ad terminum generalem inveniendum, si denominator fractionis fuerit Potestas, ut $(1 - 2pz.\cos.\Phi + ppzz)^k$, conveniet hanc fractionem resolvi in duas etsi imaginarias

$$\frac{a}{(1-(\cos.\Phi+\sqrt{-1}.\sin.\Phi)pz)^k} + \frac{b}{(1-(\cos.\Phi-\sqrt{-1}.\sin.\Phi)pz)^k},$$

quarum simul sumtarum terminus generalis Seriei ex ipsis ortæ erit

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\quad 2.\quad 3\ldots\ldots(k-1)}(\cos.n\Phi+\sqrt{-1}.\sin.n\Phi)\,ap^n z^n +$$
$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\quad 2.\quad 3\ldots\ldots(k-1)}(\cos.n\Phi-\sqrt{-1}.\sin.n\Phi)\,bp^n z^n.$$

Sit $a+b=f$; $a-b=\frac{g}{\sqrt{-1}}$, ut sit $a=\frac{f\sqrt{-1}+g}{2\sqrt{-1}}$ & $b=\frac{f\sqrt{-1}-g}{2\sqrt{-1}}$, eritque hæc expressio

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\quad 2.\quad 3\ldots\ldots(k-1)}(f.\cos.n\Phi+g.\sin.n\Phi)p^n z^n$$

terminus generalis Seriei, quæ oritur ex his fractionibus

$$\frac{\frac{1}{2}f+\frac{1}{2\sqrt{-1}}g}{(1-(\cos.\Phi+\sqrt{-1}.\sin.\Phi)pz)^k} + \frac{\frac{1}{2}f-\frac{1}{2\sqrt{-1}}g}{(1-(\cos.\Phi-\sqrt{-1}.\sin.\Phi)pz)^k},$$

seu quæ oritur ex hac fractione una

$$\frac{\begin{matrix} f-kfpz.\cos.\Phi+\frac{k(k-1)}{1.\,2}fp^2z^2.\cos.2\Phi-\frac{k(k-1)(k-2)}{1.\,2.\,3}fp^3z^3.\cos.3\Phi \\ +kgpz.\sin.\Phi-\frac{k(k-1)}{1.\,2}gp^2z^2.\sin.2\Phi+\frac{k(k-1)(k-2)}{1.\,2.\,3}gp^3z^3.\sin.3\Phi \end{matrix}\ \&c.}{(1-2pz.\cos.\Phi+ppzz)^k}$$

220. Posito ergo $k=2$, erit Seriei ex hac fractione

$$\frac{f-2pz.(f.\cos.\Phi-g.\sin.\Phi)+ppzz.(f.\cos.2\Phi-g.\sin.2\Phi)}{(1-2pz.\cos.\Phi+ppzz)^2}$$

ortæ terminus generalis $=(n+1)(f.\cos.n\Phi+g.\sin.n\Phi)p^n z^n$. At Seriei ex hac fractione $\frac{a}{1-2pz.\cos.\Phi+ppzz}$, seu hac

$a -$

$\frac{a - 2apz.\,cos.\,\phi + appzz}{(1 - 2pz.\,cos.\,\phi + ppzz)^2}$ ortæ terminus generalis est $=$ $\frac{a\,sin.\,(n+1)\,\phi}{sin.\,\phi}\,p^n z^n$. Addantur hæ fractiones invicem, ac ponatur $a + f = A$; $2a.cos.\phi + 2f.cos.\phi - 2g.\,sin.\phi = - B$ & $a + f.\,cos.\,2\phi - g.\,sin.\,2\phi = 0$, hinc erit $g = \frac{B + 2A\,cos.\,\phi}{2\,sin.\,\phi}$, $a = \frac{A + B\,cos.\,\phi}{1 - cos.\,2\phi} = \frac{A + B\,cos.\,\phi}{2\,(sin.\,\phi)^2}$ & $f = - \frac{A\,cos.\,2\phi - B\,cos.\,\phi}{2\,(sin.\,\phi)^2}$, & $g = \frac{B\,sin.\,\phi + A\,sin.\,2\phi}{2\,(sin.\,\phi)^2}$. Hanc ob rem Seriei ex hac fractione $\frac{A + Bpz}{(1 - 2pz.\,cos.\,\phi + ppzz)^2}$, ortæ terminus generalis est $\frac{A + B\,cos.\,\phi}{2\,(sin.\,\phi)^3}\,sin.\,(n+1)\,\phi.p^n z^n + (n+1)$ $\frac{(B\,sin.\,\phi.\,sin.n\phi + A\,sin.2\phi.\,sin.n\phi - B\,cos.\phi.\,cos.n\phi - A\,cos.2\phi.\,cos.n\phi)}{2\,(sin.\,\phi)^2}$ $p^n z^n = - \frac{(n+1)\,(A\,cos.(n+2)\phi + B\,cos.(n+1)\phi)}{2\,(sin.\,\phi)^2}\,p^n z^n +$ $\frac{(A + B\,cos.\,\phi)\,sin.\,(n+1)\,\phi}{2\,(sin.\,\phi)^3}\,p^n z^n =$

$$\frac{(\frac{1}{2}(n+3)\,sin.\,(n+1)\phi - \frac{1}{2}(n+1)\,sin.\,(n+3)\,\phi)}{2\,(sin.\,\phi)^3}\,A\,p^n z^n +$$

$$\frac{(\frac{1}{2}(n+2)\,sin.\,n\phi - \frac{1}{2}\,n\,sin.\,(n+2)\,\phi)}{2\,(sin.\,\phi)^3}\,B\,p^n z^n.$$ Est ergo iste terminus generalis quæsitus $=$

$$\frac{(n+3)\,sin.\,(n+1)\,\phi - (n+1)\,sin.\,(n+3)\,\phi}{4\,(sin.\phi)^3}\,A\,p^n z^n +$$

$\frac{(n+2)\,sin.n\phi - n.\,sin.\,(n+2)\,\phi}{4\,(sin.\phi)^3}\,B\,p^n z^n$: Seriei quæ oritur ex fractione $\frac{A + Bpz}{(1 - 2pz.cos.\,\phi + ppzz)^2}$.

221. Sit $k = 3$, eritque Seriei ex hac fractione ortæ

$$\frac{f - 3pz.(f.cos.\phi - g.sin.\phi) + 3ppzz\,(f.cos.2\phi - g.sin.2\phi) - p^3 z^3\,(f.cos.3\phi - g.sin.3\phi)}{(1 - 2pz.cos.\phi + ppzz)^3}$$

terminus generalis $= \frac{(n+1)(n+2)}{1.\ \ 2}\,(f.cos.n\phi + g.sin.n\phi)\,p^n z^n$.

Deinde

LIB. I. Deinde Seriei ex fractione $\frac{a+bpz}{(1-2pz.\cos.\Phi+ppzz)^2}$, seu ex hac

$$\frac{a-2apz.\cos.\Phi+appzz+bpz-2bppzz.\cos.\Phi+bp^3z^3}{(1-2pz.\cos.\Phi+ppzz)^3}$$

ortæ terminus generalis est $\frac{(n+3)\sin.(n+1)\Phi-(n+1)\sin.(n+3)\Phi}{4(\sin.\Phi)^3}ap^nz^n+\frac{(n+2)\sin.n\Phi-n.\sin.(n+2)\Phi}{4(\sin.\Phi)^3}bp^nz^n$. Addantur hæ fractiones ac ponatur numerator $=A$, erit $a+f=A$, $3f.\cos.\Phi-3g.\sin.\Phi+2a.\cos.\Phi-b=0$, $3f.\cos.2\Phi-3g.\sin.2\Phi+a-2b.\cos.\Phi=0$; & $b=f.\cos.3\Phi-g.\sin.3\Phi$, hinc erit $a=\frac{f.\cos.3\Phi-g.\sin.3\Phi-3f.\cos.\Phi+3g.\sin.\Phi}{2\cos.\Phi}=2g.(\sin.\Phi)^2\,\mathrm{tang}.\Phi-f-2f.(\sin.\Phi)^2$. Deinde reperitur $\frac{f}{g}=\frac{\sin.5\Phi-2\sin.3\Phi+\sin.\Phi}{\cos.5\Phi-2\cos.3\Phi+\cos.\Phi}$ & $a+f=A=2g.(\sin.\Phi)^2\,\mathrm{tang}.\Phi-2f(\sin.\Phi)^2$; ergo $\frac{A}{2(\sin.\Phi)^2}=\frac{g\sin.\Phi-f.\cos.\Phi}{\cos.\Phi}$; ex quibus tandem oritur $f=\frac{A(\sin.\Phi-2\sin.3\Phi+\sin.5\Phi)}{16(\sin.\Phi)^5}$, $g=\frac{A(\cos.\Phi-2\cos.3\Phi+\cos.5\Phi)}{16(\sin.\Phi)^5}$, ob $16(\sin.\Phi)^5=\sin.5\Phi-5\sin.3\Phi+10\sin.\Phi$, erit $a=\frac{A(9\sin.\Phi-3\sin.3\Phi)}{16(\sin.\Phi)^5}$ & $b=\frac{A(-\sin.2\Phi+\sin.2\Phi)}{16(\sin.\Phi)^5}=0$. Est autem $3\sin.\Phi-\sin.3\Phi=4(\sin.\Phi)^3$; ergo $a=\frac{3A}{4(\sin.\Phi)^2}$. Quocirca erit terminus generalis $\frac{(n+1)(n+2)}{1.\;\;2}p^nz^n$

$$A\frac{(\sin.(n+1)\Phi-2\sin.(n+3)\Phi+\sin.(n+5)\Phi)}{16(\sin.\Phi)^5}+$$

$$3Ap^nz^n.\frac{((n+3)\sin.(n+1)\Phi-(n+1)\sin.(n+3)\Phi)}{16(\sin.\Phi)^5}=$$

$$\frac{Ap^nz^n}{16(\sin.\Phi)^5}\left(\frac{(n+4)(n+5)}{1.\;\;2}\sin.(n+1)\Phi-\frac{2(n+1)(n+5)}{1.\;\;2}\sin.(n+3)\Phi+\frac{(n+1)(n+2)}{1.\;\;2}\sin.(n+5)\Phi\right).$$

222. Se-

222. Seriei ergo quæ oritur ex hac fractione

$$\frac{A + Bpz}{(1 - 2pz.\cos.\phi + ppzz)^3}$$

terminus generalis erit hic

$$\frac{Ap^n z^n}{16(\sin.\phi)^5}\left(\frac{(n+5)(n+4)}{1.\ 2}\sin.(n+1)\phi - \frac{2(n+1)(n+5)}{1.\ 2}\times\right.$$
$$\left.\sin.(n+3)\phi + \frac{(n+1)(n+2)}{1.\ 2}\sin.(n+5)\phi\right)$$
$$+\frac{Bp^n z^n}{16(\sin.\phi)^5}\left(\frac{(n+4)(n+3)}{1.\ 2}\sin.n\phi - \frac{2n(n+4)}{1.\ 2}\sin.(n+2)\phi + \right.$$
$$\left.\frac{n(n+1)}{1.\ 2}\sin.(n+4)\phi\right).$$

Atque, ulterius progrediendo, Seriei, quæ oritur ex hac fractione

$$\frac{A + Bpz}{(1 - 2pz.\cos.\phi + ppzz)^4}$$

terminus generalis erit hic

$$+\frac{Ap^n z^n}{64(\sin.\phi)^7}\left(\frac{(n+7)(n+6)(n+5)}{1.\ 2.\ 3}\sin.(n+1)\phi - \right.$$
$$\frac{3(n+1)(n+7)(n+6)}{1.\ 2.\ 3}\sin.(n+3)\phi + \frac{3(n+1)(n+2)(n+7)}{1.\ 2.\ 3}\times$$
$$\left.\sin.(n+5)\phi - \frac{(n+1)(n+2)(n+3)}{1.\ 2.\ 3}\sin.(n+7)\phi\right)$$
$$+\frac{Bp^n z^n}{64(\sin.\phi)^7}\left(\frac{(n+6)(n+5)(n+4)}{1.\ 2.\ 3}\sin.n\phi - \right.$$
$$\frac{3n(n+6)(n+5)}{1.\ 2.\ 3}\sin.(n+2)\phi + \frac{3n(n+1)(n+6)}{1.\ 2.\ 3}$$
$$\left.\sin.(n+4)\phi - \frac{n(n+1)(n+2)}{1.\ 2.\ 3}\sin.(n+6)\phi\right).$$

Ex his autem expressionibus facile intelligitur, quemadmodum formæ terminorum generalium pro altioribus dignitatibus progrediantur. Ad naturam vero harum expressionum penitius inspiciendam notari convenit esse

LIB. I.

$$\sin.\Phi = \sin.\Phi$$
$$4(\sin.\Phi)^3 = 3\sin.\Phi - \sin.3\Phi$$
$$16(\sin.\Phi)^5 = 10\sin.\Phi - 5\sin.3\Phi + \sin.5\Phi$$
$$64(\sin.\Phi)^7 = 35\sin.\Phi - 21\sin.3\Phi + 7\sin.5\Phi - \sin.7\Phi$$
$$256(\sin.\Phi)^9 = 126\sin.\Phi - 84\sin.3\Phi + 36\sin.5\Phi - 9\sin.7\Phi + \sin.9\Phi$$
&c.

223. Cum igitur hoc pacto omnes functiones fractæ in fractiones partiales reales resolvi queant, simul omnium Serierum recurrentium termini generales per expressiones reales exhiberi poterunt. Quod quo clarius appareat, exempla sequentia adjuncta sunt.

EXEMPLUM I.

Ex fractione $\frac{1}{(1-z)(1-zz)(1-z^3)} = \frac{1}{1-z-zz+z^4+z^5-z^6}$, oritur ista Series recurrens $1+z+2z^2+3z^3+4z^4+5z^5+7z^6+8z^7+10z^8+12z^9+$&c.; cujus terminus generalis desideratur. Fractio proposita secundum Factores ordinata fit $=\frac{1}{(1-z)^3(1+z)(1+z+zz)}$, quæ resolvitur in has fractiones $\frac{1}{6(1-z)^3}+\frac{1}{4(1-z)^2}+\frac{17}{72(1-z)}+\frac{1}{8(1+z)}+\frac{(2+z)}{9(1+z+zz)}$. Harum prima $\frac{1}{6(1-z)^3}$ dat terminum generalem $\frac{(n+1)(n+2)}{1.\ \ 2.}\cdot\frac{1}{6}z^n = \frac{nn+3n+2}{12}z^n$: secunda $\frac{1}{4(1-z)^2}$ dat $\frac{n+1}{4}z^n$: tertia $\frac{17}{72(1-z)}$ dat $\frac{17}{72}z^n$: quarta $\frac{1}{8(1+z)}$ dat $\frac{1}{8}(-1)^n z^n$.

Quinta

Cap. XIII.

Quinta vero $\frac{2+z}{9(1+z+zz)}$ comparata cum forma $\frac{A+Bpz}{1-2pz.\cos.\phi+ppzz}$ (218) dat $p=1$, $\phi=\frac{\varpi}{3}=60°$; $A=+\frac{2}{9}$; & $B=-\frac{1}{9}$, unde oritur terminus generalis $+\frac{2\,\text{fin.}(n+1)\phi-\text{fin.}n\phi}{9\,\text{fin.}\phi}(-1)^n z^n=+\frac{4\,\text{fin.}(n+1)\phi-2\,\text{fin.}n\phi}{9\sqrt{3}}(-1)^n z^n=+\frac{4\,\text{fin.}(n+1)\frac{\varpi}{3}-2\,\text{fin.}n\frac{\varpi}{3}}{9\sqrt{3}}(-1)^n z^n$. Colligantur hæ expreſſiones omnes in unam ſummam, ac prodibit Seriei propoſitæ terminus generalis quæſitus $=(\frac{nn}{12}+\frac{n}{2}+\frac{47}{72})z^n\pm\frac{1}{8}z^n\pm\frac{4\,\text{fin.}(n+1)\frac{\varpi}{3}-2\,\text{fin.}n\frac{\varpi}{3}}{9\sqrt{3}}z^n$, ubi ſigna ſuperiora valent ſi n numerus par, inferiora ſin impar. Ubi notandum eſt ſi fuerit n numerus formæ $3m$ fore $\frac{4\,\text{fin.}\frac{1}{3}(n+1)\varpi-2\,\text{fin.}\frac{1}{3}n\varpi}{9\sqrt{3}}=\pm\frac{2}{9}$; ſi fuerit $n=3m+1$ erit hæc expreſſio $=\mp\frac{1}{9}$; at ſi $n=3m+2$ erit iſta expreſſio $=\mp\frac{1}{9}$, prout n fuerit numerus vel par vel impar. Ex his natura Seriei ita explicari poteſt, ut

si fuerit	terminus generalis futurus sit
$n = 6m + 0$	$(\frac{nn}{12} + \frac{n}{2} + 1)z^n$
$n = 6m + 1$	$(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12})z^n$
$n = 6m + 2$	$(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3})z^n$
$n = 6m + 3$	$(\frac{nn}{12} + \frac{n}{2} + \frac{3}{4})z^n$
$n = 6m + 4$	$(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3})z^n$
$n = 6m + 5$	$(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12})z^n$

Sic, si fuerit $n = 50$, valet forma $n = 6m + 2$, eritque terminus Seriei $= 234z^{50}$

EXEMPLUM II.

Ex fractione $\frac{1 + z + zz}{1 - z - z^4 + z^5}$, oritur hæc Series recurrens $1 + 2z + 3zz + 3z^3 + 4z^4 + 5z^5 + 6z^6 + 6z^7 + 7z^8 + \&c.$, cujus terminum generalem invenire oportet. Fractio proposita ad hanc formam reducitur $\frac{1 + z + zz}{(1 - z)^2(1 + z)(1 + zz)}$, quæ propterea resolvitur in has fractiones partiales $\frac{3}{4(1 - z)^2} + \frac{3}{8(1 - z)} + \frac{1}{8(1 + z)} - \frac{1 + z}{4(1 + zz)}$. Harum prima $\frac{3}{4(1 - z)^2}$ dat terminum generalem $\frac{3(n + 1)}{4} z^n$; secunda $\frac{3}{8(1 - z)}$ dat $\frac{3}{8} z^n$; tertia dat $\frac{1}{8}(-1)^n z^n$; & quarta $- \frac{1 + z}{4(1 + zz)}$ comparata cum forma $\frac{A + Bpz}{1 - 2pz.\cos.\phi + ppzz}$. dat $p = 1$; $\cos.\phi = 0$; & $\phi = \frac{1}{2}\varpi$; $A = -\frac{1}{4}$;

B =

$B = +\frac{1}{4}$, unde fit terminus generalis $= (-\frac{1}{4}\,\mathit{fin.}\,\frac{1}{2}(n+1)\varpi + \frac{1}{4}\,\mathit{fin.}\,\frac{1}{2}\,n\varpi)\,z^n$. Quare colligendo erit terminus generalis quæſitus $= (\frac{3}{4}n + \frac{9}{8})\,z^n \pm \frac{1}{8}\,z^n - \frac{1}{4}\,(\mathit{fin.}\,\frac{1}{2}(n+1)\varpi - \mathit{fin.}\,\frac{1}{2}\,n\varpi)\,z^n$. Hinc CAP. XIII.

ſi fuerit	erit terminus generalis
$n = 4m + 0$	$(\frac{3}{4}n + 1)\,z^n$
$n = 4m + 1$	$(\frac{3}{4}n + \frac{5}{4})\,z^n$
$n = 4m + 2$	$(\frac{3}{4}n + \frac{3}{2})\,z^n$
$n = 4m + 3$	$(\frac{3}{4}n + \frac{3}{4})\,z^n$

Ita, ſi $n = 50$, valebit $n = 4m + 2$, eritque terminus $= 39z^{50}$.

224. Propoſita ergo Serie recurrente, quoniam illa fractio unde oritur, facile cognoſcitur, ejus terminus generalis ſecundum præcepta data reperietur. Ex lege autem Seriei recurrentis, qua quiſque terminus ex præcedentibus definitur, ſtatim innoteſcit denominator fractionis, hujuſque Factores præbebunt formam termini generalis, per numeratorem enim tantum coefficientes determinantur. Sit nempe propoſita hæc Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus lex progreſſionis, qua unuſquiſque terminus ex aliquot præcedentibus determinatur, præbeat hunc fractionis denominatorem $1 - \alpha z - \beta z^2 - \gamma z^3$. Ita ut ſit $D = \alpha C + \beta B + \gamma A$; $E = \alpha D + \beta C + \gamma B$; $F = \alpha E + \beta D + \gamma C$; &c.,

LIB. I. &c., qui multiplicatores $\alpha, + \beta, + \gamma$ a MOIVRÆO *ſcalam relationis* conſtituere dicuntur. Lex ergo progreſſionis poſita eſt in ſcala relationis, atque ſcala relationis ſtatim præbet denominatorem fractionis, ex cujus reſolutione propoſita Series recurrens oritur.

225. Ad terminum ergo generalem, ſeu coëfficientem Poteſtatis indefinitæ z^n, inveniendum, quæri debent denominatoris $1 - \alpha z - \beta z^2 - \gamma z^3$ Factores vel ſimplices vel duplices, ſi imaginarios vitare velimus. Sint primo Factores ſimplices omnes inter ſe inæquales & reales hi $(1 - pz)(1 - qz)(1 - rz)$; atque fractio generans Seriem propoſitam reſolvetur in $\frac{A}{1 - pz} + \frac{B}{1 - qz} + \frac{C}{1 - rz}$; unde Seriei terminus generalis erit $(Ap^n + Bq^n + Cr^n)z^n$. Si duo Factores fuerint æquales nempe $q = p$, tum terminus generalis hujuſmodi erit $((An + B)p^n + Cr^n)z^n$, &, ſi inſuper fuerit $r = q = p$, erit terminus generalis $(An^2 + Bn + C)p^n z^n$. Quod ſi vero denominator $1 - \alpha z - \beta z^2 - \gamma z^3$ duplicem habeat Factorem, ut ſit $= (1 - pz)(1 - 2qz.\mathit{coſ}.\phi + qqzz)$ tum terminus generalis erit $= (Ap^n + \frac{B\,\mathit{ſin}.(n+1)\phi + C\,\mathit{ſin}.n\phi}{\mathit{ſin}.\phi} q^n)z^n$. Cum igitur, poſitis pro n ſucceſſive numeris 0, 1, 2, prodire debeant termini A, Bz, Cz^2, hinc valores litterarum A, B, C determinabuntur.

226. Sit ſcala relationis bimembris, ſeu determinetur quiſque terminus per duos præcedentes, ita ut ſit

$$C = \alpha B - \beta A;\ D = \alpha C - \beta B;\ E = \alpha D - \beta C, \text{ \&c.},$$

atque manifeſtum eſt Seriem hanc recurrentem, quæ ſit

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots\ldots + Pz^n + Qz^{n+1} + \text{\&c.},$$

oriri ex fractione cujus denominator ſit $1 - \alpha z + \beta zz$. Sint hujus denominatoris Factores $(1 - pz)(1 - qz)$ erit $p +$

$q = \alpha$ & $pq = \beta$: atque Seriei terminus generalis erit $(Ap^n + Bq^n)z^n$. Hinc facto $n = 0$, erit $A = A + B$; & facto $n = 1$ erit $A = Ap + Bq$; unde fit $Aq - B = A(q-p)$ & $A = \frac{Aq - B}{q - p}$; & $B = \frac{Ap - B}{p - q}$. Inventis autem valoribus A & B, erit $P = Ap^n + Bq^n$ & $Q = Ap^{n+1} + Bq^{n+1}$. Tum vero erit $AB = \frac{BB - \alpha AB + \beta AA}{4\beta - \alpha\alpha}$.

Cap. XIII.

227. Hinc deduci potest modus quemvis terminum ex unico præcedente formandi, cum ad hoc per legem progressionis duo requirantur. Cum enim sit

$$P = Ap^n + Bq^n \text{ \& } Q = Ap.p^n + Bq.q^n$$

erit

$$Pq - Q = A(q-p)p^n \text{ \& } Pp - Q = B(p-q)q^n:$$

multiplicentur hæ expressiones in se invicem; eritque

$$P^2pq - (p+q)PQ + QQ - AB(p-q)^2 p^n q^n = 0.$$

At est

$p + q = \alpha$; $pq = \beta$; $(p-q)^2 = (p+q)^2 - 4pq = \alpha\alpha - 4\beta$ & $p^n q^n = \beta^n$. Quibus substitutis habebitur

$\beta P^2 - \alpha PQ + QQ = (\beta AA - \alpha AB + BB)\beta^n$, seu $\frac{QQ - \alpha PQ + \beta PP}{BB - \alpha AB + \beta AA} = \beta^n$; quæ est insignis proprietas Serierum recurrentium, quarum quisque terminus per duos præcedentes determinatur. At cognito quovis termino P, erit sequens $Q = \frac{1}{2}\alpha P + \sqrt{((\frac{1}{4}\alpha^2 - \beta)P^2 + (B^2 - \alpha AB + \beta AA)\beta^n)}$, quæ expressio, etsi speciem irrationalitatis præ se fert,

LIB. I. fert, tamen semper est rationalis, propterea quod termini irrationales in Serie non occurrunt.

228. Ex datis porro duobus terminis contiguis quibusvis Pz^n & Qz^{n+1} commode assignari potest terminus multo magis remotus Xz^{2n} Ponatur enim

$X = fP^2 + gPQ - hAB\mathfrak{C}^n$ Quoniam est

$P = Ap^n + Bq^n$ & $Q = Ap.p^n + Bq.q^n$ atque

$X = Ap^{2n} + Bq^{2n}$; erit ut sequitur

$$\begin{aligned} fP^2 &= fA^2p^{2n} + fB^2q^{2n} + 2fAB\mathfrak{C}^n \\ gPQ &= gA^2p.p^{2n} + gB^2q.q^{2n} + gAB\alpha\mathfrak{C}^n \\ -hAB\mathfrak{C}^n &= \qquad\qquad\qquad\qquad\qquad - hAB\mathfrak{C}^n \\ \hline X &= Ap^{2n} + Bq^{2n} \end{aligned}$$

Fiet ergo $f + gp = \frac{1}{A}$; $f + gq = \frac{1}{B}$ & $h = 2f + g\alpha$, unde $g = \frac{B - A}{AB(p - q)}$ & $f = \frac{Ap - Bq}{AB(p - q)}$. At est $B - A = \frac{\alpha A - 2B}{p - q}$; $Ap - Bq = \frac{\alpha B - 2A\mathfrak{C}}{p - q}$. Ergo $f = \frac{\alpha B - 2A\mathfrak{C}}{AB(\alpha\alpha - 4\mathfrak{C})}$ & $g = \frac{\alpha A - 2B}{AB(\alpha\alpha - 4\mathfrak{C})}$ seu $f = \frac{2A\mathfrak{C} - \alpha B}{BB - \alpha AB + \mathfrak{C}AA}$ & $g = \frac{2B - \alpha A}{BB - \alpha AB + \mathfrak{C}AA}$; ideoque $h = \frac{(4\mathfrak{C} - \alpha\alpha)A}{BB - \alpha AB + \mathfrak{C}AA}$.

Eritque ergo

$$X = \frac{(2A\mathfrak{C} - \alpha B)P^2 + (2B - \alpha A)PQ}{BB - \alpha AB + \mathfrak{C}AA} - A\mathfrak{C}^n.$$

Simili vero modo reperitur

$$X = \frac{(\alpha\mathfrak{C}A - (\alpha\alpha - 2\mathfrak{C})B)P^2 + (2B - \alpha A)Q^2}{\alpha(BB - \alpha AB + \mathfrak{C}AA)} - \frac{2B\mathfrak{C}^n}{\alpha}.$$

His conjungendis per eliminationem termini $\mathfrak{C}^n$ reperitur

$$X = \frac{(\mathfrak{C}A - \alpha B)P^2 + 2BPQ - AQQ}{BB - \alpha AB + \mathfrak{C}AA}.$$

229. Si

229. Simili modo, si statuantur termini sequentes CAP. XIII.

$A + Bz + Cz^2 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \ldots\ldots +$

$Xz^{2n} + Yz^{2n+1} + Zz^{2n+2}$,

erit

$$Z = \frac{(\beta A - \alpha B)Q^2 + 2BQR - ARR}{BB - \alpha AB + \beta AA}, \text{ \&, ob } R = \alpha Q - \beta P,$$

erit

$$Z = \frac{-\beta\beta AP^2 + 2\beta(\alpha A - B)PQ + (\alpha B - (\alpha\alpha - \beta)A)Q^2}{BB - \alpha AB + \beta AA}$$

At est

$Z = \alpha Y - \beta X$, ergo $Y = \frac{Z + \beta X}{\alpha}$; unde fit

$Y = \frac{-\beta BP^2 + 2\beta APQ + \alpha(B - \alpha A)QQ}{BB - \alpha AB + \beta AA}$. Sic igitur porro ex X & Y definiri poterunt simili modo coëfficientes potestatum z^{4n}, & z^{4n+1}; hincque ipsarum z^{8n}, z^{8n+1}, & ita porro.

EXEMPLUM.

Sit proposita ista Series recurrens

$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \ldots\ldots + Pz^n + Qz^{n+1} + \text{\&c.}$,

cujus cum quilibet coëfficiens sit summa duorum præcedentium, erit denominator fractionis hanc Seriem producentis $1 - z - zz$; ideoque $\alpha = 1$; $\beta = -1$; & $A = 1$; $B = 3$; unde fit $BB - \alpha AB + \beta AA = 5$, ex quo orietur primum $Q = \frac{P + \surd(5PP + 20(-1)^n)}{2} = \frac{P + \surd(5PP \pm 20)}{2}$, ubi signum superius valet, si n sit numerus par, inferius si impar. Sic, si $n = 4$, ob $P = 11$, erit

LIB. I. $Q = \frac{11 + \surd(5.121 + 20)}{2} = \frac{11 + 25}{2} = 18$. Si porro coëfficiens termini z^{2n} sit X, erit $X = \frac{-4PP + 6PQ - QQ}{5}$; ergo Poteſtatis z^8 coëfficiens erit $= \frac{-4.121 + 6.198 - 324}{5} = 76$. Cum autem ſit $Q = \frac{P + \surd(5PP \pm 20)}{2}$ erit $QQ = \frac{3PP \pm 10 + P\surd(5PP \pm 20)}{2}$; ideoque $X = \frac{-PP \mp 2 + P\surd(5PP \pm 20)}{2}$

Ex termino ergo Seriei quocunque Pz^n, obtinentur hi $\frac{P + \surd(5PP \pm 20)}{2} z^{n+1}$, & $\frac{-PP \mp 2 + P\surd(5PP \pm 20)}{2} z^{2n}$.

230. Simili modo in Seriebus recurrentibus, quarum quilibet terminus ex tribus antecedentibus determinatur, quivis terminus ex duobus antecedentibus definiri poteſt. Sit enim Series hujuſmodi recurrens

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \&c.,$$

cujus ſcala relationis ſit $\alpha, -\beta, +\gamma$, ſeu quæ oriatur ex fractione cujus denominator $= 1 - \alpha z + \beta z^2 - \gamma z^3$. Quod ſi jam termini P, Q, R eodem modo per Factores hujus denominatoris, qui ſint $(1 - pz)(1 - qz)(1 - rz)$ exprimantur, ut ſit $P = \mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n$; $Q = \mathfrak{A}p.p^n + \mathfrak{B}q.q^n + \mathfrak{C}r.r^n$; & $R = \mathfrak{A}p^2.p^n + \mathfrak{B}q^2.q^n + \mathfrak{C}r^2.r^n$; ob $p + q + r = \alpha$; $pq + pr + qr = \beta$ & $pqr = \gamma$, reperietur hæc proportio

$$R^3 \left.\begin{matrix} -2\alpha Q \\ +\beta P \end{matrix}\right\} R^2 \left.\begin{matrix} +(\alpha\alpha + \beta)\,Q^2 \\ -(\alpha\beta + 3\gamma)PQ \\ +\alpha\gamma\quad P^2 \end{matrix}\right\} R \begin{matrix} -(\alpha\beta - \gamma)\,Q^3 \\ +(\alpha\gamma + \beta\beta)PQ^2 \\ -\quad 2\beta\gamma\quad P^2Q \\ +\quad \gamma\gamma\quad P^3 \end{matrix} : c^n =$$

C³

$$C^3 \left. \begin{matrix} -2\alpha B \\ +\beta A \end{matrix} \right\} C^2 \left. \begin{matrix} +(\alpha^2+\beta)B^2 \\ -(\alpha\beta+3\gamma)AB \\ +\alpha\gamma\quad A^2 \end{matrix} \right\} C \begin{matrix} -(\alpha\beta-\gamma)B^3 \\ +(\alpha\gamma+\beta\beta)AB^2 \\ -\quad 2\beta\gamma A^2B \\ +\quad \gamma\gamma A^3 \end{matrix} : 1.$$

Pendet ergo inventio termini R ex duobus præcedentibus P & Q a resolutione æquationis cubicæ.

231. His de terminis generalibus Serierum recurrentium notatis, superest ut earumdem Serierum summas investigemus. Ac primo quidem manifestum est summam Seriei recurrentis in infinitum extensæ æqualem esse fractioni ex qua oritur: cujus fractionis cum denominator ex ipsa progressionis lege pateat, reliquum est ut numeratorem definiamus. Sit itaque proposita hæc Series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \&c.,$$

cujus lex progressionis præbeat hunc denominatorem $1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4$. Sumamus fractionem summæ Seriei in infinitum æqualem esse $= \dfrac{a + bz + cz^2 + dz^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$, ex qua cum Series proposita oriri debeat, erit comparando

$$\begin{aligned} a &= A \\ b &= B - \alpha A \\ c &= C - \alpha B + \beta A \\ d &= D - \alpha C + \beta B - \gamma A: \end{aligned}$$

Hinc erit summa quæsita

$$\frac{A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(D - \alpha C + \beta B - \gamma A)z^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

232. Hinc facile intelligitur quemadmodum Seriei recurrentis summa ad datum terminum usque inveniri debeat.

 Quæ-

LIB I. Quæratur ſcilicet Seriei modo aſſumtæ ſumma ad terminum Pz^n, atque ponatur

$$s = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots\ldots + Pz^n,$$

quoniam hujus Seriei ſumma in infinitum conſtat, quæratur ſumma terminorum ultimum Pz^n in infinitum ſequentium, qui ſint

$$t = Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + Tz^{n+4} + \&c.,$$

hæc Series per z^{n+1} diviſa dat Seriem recurrentem propoſitæ æqualem, cujus propterea ſumma erit $t =$

$$\frac{Qz^{n+1} + (R - \alpha Q)z^{n+2} + (S - \alpha R + \beta Q)z^{n+3} +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

Unde orietur ſumma quæſita $s =$

$$\frac{+ A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(D - \alpha C + \beta B - \gamma A)z^3 - Qz^{n+1}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4} -$$

$$\frac{(R - \alpha Q)z^{n+2} - (S - \alpha R + \beta Q)z^{n+3}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4} -$$

$$\frac{(T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

233. Quod ſi ergo ſcala relationis fuerit bimembris

α, —

α, $-\beta$; Seriei $A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n$, quæ oritur ex fractione $\frac{A + (B - \alpha A)z}{1 - \alpha z + \beta zz}$, ſumma erit Cap. XIII.

$$\frac{A + (B - \alpha A)z - Qz^{n+1} - (R - \alpha Q)z^{n+2}}{1 - \alpha z + \beta zz}$$

At eſt, ex natura Seriei, $R = \alpha Q - \beta P$, unde prodibit ſumma

$$\frac{A + (B - \alpha A)z - Qz^{n+1} + \beta Pz^{n+2}}{1 - \alpha z + \beta zz}$$

Exemplum.

Sit propoſita Series $1 + 3z + 4z^2 + 7z^3 + \ldots\ldots + Pz^n$ ubi eſt $\alpha = 1$; $\beta = -1$; $A = 1$; $B = 3$; erit hujus ſumma

$$\frac{1 + 2z - Qz^{n+1} - Pz^{n+2}}{1 - z - zz}.$$ Poſito vero $z = 1$, erit ſumma Seriei $1 + 3 + 4 + 7 + 11 + \ldots\ldots\ldots + P = P + Q - 3$. Summa ergo termini ultimi & ſequentis ternario excedit ſummam Seriei. Quia vero eſt $Q = \frac{P + \surd(5PP \pm 20)}{2}$ erit ſumma Seriei $1+3+4+7+11+\ldots\ldots+P = \frac{3P - 6 + \surd(5PP \pm 20)}{2}$. Ex ſolo ergo termino ultimo ſumma poteſt exhiberi.

CAPUT XIV.

De multiplicatione ac divisione Angulorum.

234. Sit Angulus, vel Arcus, in Circulo cujus Radius $= 1$, quicunque $= z$; ejus Sinus $= x$; Cosinus $= y$, & Tangens $= t$; erit $xx + yy = 1$ & $t = \frac{x}{y}$. Cum igitur, uti supra vidimus; tam Sinus quam Cosinus Angulorum z; $2z$; $3z$; $4z$; $5z$; &c., constituant Seriem recurrentem cujus scala relationis est $2y, -1$; primum Sinus horum Arcuum ita se habebunt:

$$\begin{aligned}
\textit{fin.}\,0z &= 0 \\
\textit{fin.}\,1z &= x \\
\textit{fin.}\,2z &= 2xy \\
\textit{fin.}\,3z &= 4xy^2 - x \\
\textit{fin.}\,4z &= 8xy^3 - 4xy \\
\textit{fin.}\,5z &= 16xy^4 - 12xy^2 + x \\
\textit{fin.}\,6z &= 32xy^5 - 32xy^3 + 6xy \\
\textit{fin.}\,7z &= 64xy^6 - 80xy^4 + 24xy^2 - x \\
\textit{fin.}\,8z &= 128xy^7 - 192xy^5 + 80xy^3 - 0xy
\end{aligned}$$

hinc concluditur fore

$$\textit{fin.}\,nz = x\Big(2^{n-1}y^{n-1} - (n-2)2^{n-3}y^{n-3} + \frac{(n-3)(n-4)}{1.\;2}2^{n-5}y^{n-5} - \frac{(n-4)(n-5)(n-6)}{1.\;2.\;3}2^{n-7}y^{n-7} + \frac{(n-5)(n-6)n-7)(n-8)}{1.\;2.\;3.\;4}2^{n-9}y^{n-9} - \&c.\Big)$$

235. Si ponamus Arcum $nz = s$; erit $\textit{fin.}\,nz = \textit{fin.}\,s = \textit{fin.}\,(\varpi - s) = \textit{fin.}\,(2\varpi + s) = \textit{fin.}\,(3\varpi - s)$ &c., hi enim

enim Sinus omnes sunt inter se æquales. Hinc obtinemus plures valores pro x, qui erunt CAP. XIV.

$$\text{fin.}\ \frac{s}{n};\ \text{fin.}\ \frac{\varpi - s}{n};\ \text{fin.}\ \frac{2\pi + s}{n};\ \text{fin.}\ \frac{3\pi - s}{n};\ \text{fin.}\ \frac{4\varpi + s}{n};\ \&c.,$$

qui ergo omnes æquationi inventæ æque conveniunt. Tot autem prodibunt diversi pro x valores, quot numerus n continet unitates, qui propterea erunt radices æquationis inventæ. Cavendum ergo est, ne valores æquales pro iisdem habeantur, quod fiet dum alternæ tantum expressiones assumantur. Cognitis igitur radicibus æquationis a posteriori, earum comparatio cum terminis æquationis notatu dignas præbebit proprietates. Quoniam autem ad hoc æquatio, in qua tantum x tanquam incognita insit, requiritur, pro y suus valor $\surd(1 - xx)$ substitui debet; unde duplex operatio instituenda erit, prout n fuerit vel numerus par vel impar.

236. Sit n numerus impar, quia Arcuum $-z, +z, +3z, +5z$; &c., differentia est $2z$, hujusque Cosinus $= 1 - 2xx$, erit progressionis Sinuum scala relationis hæc $2 - 4xx, - 1$. Hinc erit

$$\begin{aligned}
\text{fin.} -z &= -x \\
\text{fin.}\ z &= x \\
\text{fin.}\ 3z &= 3x - 4x^3 \\
\text{fin.}\ 5z &= 5x - 20x^3 + 16x^5 \\
\text{fin.}\ 7z &= 7x - 56x^3 + 112x^5 - 64x^7 \\
\text{fin.}\ 9z &= 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9
\end{aligned}$$

ergo

$$\text{fin.}\ nz = nx - \frac{n(nn-1)}{1.\ 2.\ 3}x^3 + \frac{n(nn-1)(nn-9)}{1.\ 2.\ 3.\ 4.\ 5}x^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7}x^7 + \&c.,$$

si quidem n fuerit numerus impar. Hujusque æquationis radices sunt

LIB. I. sunt $\sin. z$; $\sin. \left(\frac{2\varpi}{n}+z\right)$; $\sin. \left(\frac{4\varpi}{n}+z\right)$; $\sin. \left(\frac{6\varpi}{n}+z\right)$; $\sin. \left(\frac{8\varpi}{n}+z\right)$; &c., quarum numerus est n.

237. Hujus ergo æquationis

$$0 = 1 - \frac{nx}{\sin. nz} + \frac{n(nn-1)x^3}{1.\ 2.\ 3 \sin. nz} - \frac{n(nn-1)(nn-9)x^5}{1.\ 2.\ 3.\ 4.\ 5\ \sin. nz} \cdots \pm \frac{2^{n-1}x^n}{\sin. nz},$$

(ubi signum superius valet si n unitate deficiat a multiplo quaternarii, contra inferius,) Factores sunt $\left(1 - \frac{x}{\sin. z}\right)$ $\left(1 - \frac{x}{\sin. \left(\frac{2\varpi}{n}+z\right)}\right)\left(1 - \frac{x}{\sin. \left(\frac{4\pi}{n}+z\right)}\right)$ &c., ex quibus concluditur fore

$$\frac{n}{\sin. nz} = \frac{1}{\sin. z} + \frac{1}{\sin. \left(\frac{2\varpi}{n}+z\right)} + \frac{1}{\sin. \left(\frac{4\varpi}{n}+z\right)} + \frac{1}{\sin. \left(\frac{6\varpi}{n}+z\right)} +$$

&c., donec habeantur n termini. Tum vero productum omnium erit $\mp \frac{2^{n-1}}{\sin. nz} = \frac{1}{\sin. z. \sin. \left(\frac{2\varpi}{n}+z\right). \sin. \left(\frac{4\varpi}{n}+z\right). \sin. \left(\frac{6\varpi}{n}+z\right) \&c.}$

seu $\sin. nz = \mp 2^{n-1} \sin. z. \sin. \left(\frac{2\varpi}{n}+z\right). \sin. \left(\frac{4\varpi}{n}+z\right) \times \sin. \left(\frac{6\varpi}{n}+z\right)$ &c.. Et, quia terminus penultimus deest, erit

$$0 = \sin. z + \sin. \left(\frac{2\varpi}{n}+z\right) + \sin. \left(\frac{4\pi}{n}+z\right) + \sin. \left(\frac{6\pi}{n}+z\right) \&c.$$

EXEMPLUM I.

Si ergo fuerit $n = 3$, prodibunt hæ æqualitates
$0 = \sin. z + \sin. (120^\circ + z) + \sin. (240^\circ + z) = \sin. z + \sin. (60 - z) - \sin. (60 + z)$.

$\frac{3}{\sin. 3z}$

$$\frac{3}{\sin.3z} = \frac{1}{\sin.z} + \frac{1}{\sin.(120+z)} + \frac{1}{\sin.(240+z)} = \frac{1}{\sin.z} + \frac{1}{\sin.(60-z)} - \frac{1}{\sin.(60+z)}$$

$$\sin.3z = -4\sin.z.\sin.(120+z)\sin.(240+z) = 4\sin.z.\sin.(60-z).\sin.(60+z).$$

Erit ergo, uti jam supra notavimus,

$$\sin.(60+z) = \sin.z + \sin.(60-z), \&$$

$$3\,\mathrm{cosec}.3z = \mathrm{cosec}.z + \mathrm{cosec}.(60-z) - \mathrm{cosec}.(60+z).$$

EXEMPLUM II.

Ponamus esse $n = 5$, atque prodibunt hæ æquationes:

$$0 = \sin.z + \sin.(\tfrac{2}{5}\pi + z) + \sin.(\tfrac{4}{5}\pi + z) + \sin.(\tfrac{6}{5}\pi + z) + \sin.(\tfrac{8}{5}\pi + z)$$

seu $0 = \sin.z + \sin.(\tfrac{2}{5}\pi + z) + \sin.(\tfrac{1}{5}\pi - z) - \sin.(\tfrac{1}{5}\pi + z) - \sin.(\tfrac{2}{5}\pi - z)$

seu $0 = \sin.z + \sin.(\tfrac{1}{5}\pi - z) - \sin.(\tfrac{1}{5}\pi + z) + \sin.(\tfrac{2}{5}\pi + z) - \sin.(\tfrac{2}{5}\pi - z)$

deinde erit

$$\frac{5}{\sin.5z} = \frac{1}{\sin.z} + \frac{1}{\sin.(\frac{1}{5}\pi - z)} - \frac{1}{\sin.(\frac{1}{5}\pi + z)} - \frac{1}{\sin.(\frac{2}{5}\pi - z)} + \frac{1}{\sin.(\frac{2}{5}\pi + z)}$$

$$\sin.5z = 16\sin.z.\sin.(\tfrac{1}{5}\pi - z).\sin.(\tfrac{1}{5}\pi + z)\times\sin.(\tfrac{2}{5}\pi - z).\sin.(\tfrac{2}{5}\pi + z)$$

LIB. I.

EXEMPLUM III.

Hoc modo, si ponamus $n = 2m + 1$, erit

$$0 = \sin. z + \sin. \left(\frac{\varpi}{n} - z\right) - \sin. \left(\frac{\varpi}{n} + z\right) - \sin. \left(\frac{2\varpi}{n} - z\right) + \sin. \left(\frac{2\varpi}{n} + z\right) + \sin. \left(\frac{3\varpi}{n} - z\right) - \sin. \left(\frac{3\varpi}{n} + z\right) - \ldots \pm \sin. \left(\frac{m}{n}\pi - z\right) \mp \sin. \left(\frac{m}{n}\pi + z\right)$$

ubi signa superiora valent si m sit numerus impar, inferiora si sit par. Altera æquatio erit hæc.

$$\frac{n}{\sin. nz} = \frac{1}{\sin. z} + \frac{1}{\sin. \left(\frac{\pi}{n} - z\right)} - \frac{1}{\sin. \left(\frac{\pi}{n} + z\right)} - \frac{1}{\sin. \left(\frac{2\varpi}{n} - z\right)} + \frac{1}{\sin. \left(\frac{2\varpi}{n} + z\right)} + \frac{1}{\sin. \left(\frac{3\varpi}{n} - z\right)} - \frac{1}{\sin. \left(\frac{3\varpi}{n} + z\right)} - \ldots \pm \frac{1}{\sin. \left(\frac{m\pi}{n} - z\right)} \mp \frac{1}{\sin. \left(\frac{m\varpi}{n} + z\right)}$$

quæ ad Cosecantes commode transfertur. Tertio habetur hoc productum:

$$\sin. nz = 2^{2m} \sin. z . \sin. \left(\frac{\pi}{n} - z\right) . \sin. \left(\frac{\varpi}{n} + z\right) . \sin. \left(\frac{2\varpi}{n} - z\right) \times \sin. \left(\frac{2\varpi}{n} + z\right) . \sin. \left(\frac{3\varpi}{n} - z\right) . \sin. \left(\frac{3\varpi}{n} + z\right) . \ldots \ldots \times \sin. \left(\frac{m\pi}{n} - z\right) . \sin. \left(\frac{m\pi}{n} + z\right).$$

238. Sit n nunc numerus par, & quoniam est $y = \sqrt{(1 - xx)}$ & $\cos. 2z = 1 - 2xx$, ita ut Seriei Sinuum sit scala relationis, ut ante, $2 - 4xx, - 1$, erit

sin.

$$\text{fin. } 0z = 0$$
$$\text{fin. } 2z = 2x\sqrt{(1-xx)}$$
$$\text{fin. } 4z = (4x - 8x^3)\sqrt{(1-xx)}$$
$$\text{fin. } 6z = (6x - 32x^3 + 32x^5)\sqrt{(1-xx)}$$
$$\text{fin. } 8z = (8x - 80x^3 + 192x^5 - 128x^7)\sqrt{(1-xx)}$$

& generaliter

$$\text{fin. } nz = \left(nx - \frac{n(nn-4)}{1.\ 2.\ 3}x^3 + \frac{n(nn-4)(nn-16)}{1.\ 2.\ 3.\ 4.\ 5}x^5 - \frac{n(nn-4)(nn-16)(nn-36)}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7}x^7 + \ldots \pm 2^{n-1}x^{n-1}\right)\sqrt{(1-xx)}$$

denotante n numerum quemcunque parem.

239. Ad æquationem hanc rationalem efficiendam sumantur utrinque quadrata, ac prodibit hujusmodi æquatio

$$(\text{fin. } nz)^2 = nnxx + Px^4 + Qx^6 + \ldots - 2^{2n-2}x^{2n}$$

$$\text{seu } x^{2n} \ldots\ldots - \frac{nn}{2^{2n-2}}xx + \frac{1}{2^{2n-2}}(\text{fin. } nz)^2 = 0$$

cujus æquationis radices erunt tam affirmativæ quam negativæ; Scilicet $+\text{fin. } z$; $\pm\text{fin. }\left(\frac{\pi}{n} - z\right)$; $\pm\text{fin. }\left(\frac{2\varpi}{n} + z\right)$; $\pm\text{fin. }\left(\frac{3\varpi}{n} - z\right)$; $\pm\text{fin. }\left(\frac{4\varpi}{n} + z\right)$ &c. Sumendo omnino n hujusmodi expressiones. Cum igitur ultimus terminus sit productum omnium harum radicum, extrahendo utrinque radicem quadratam erit

$$\text{fin. } nz = \pm 2^{n-1}\text{fin. } z.\text{fin. }\left(\frac{\varpi}{n} - z\right).\text{fin. }\left(\frac{2\varpi}{n} + z\right) \times \text{fin. }\left(\frac{3\varpi}{n} - z\right)\ldots\ldots;$$

ubi, quibus casibus utrumvis signum valeat, ex casibus particularibus erit dispiciendum.

Exemplum.

Substituendo autem pro n successive numeros 2, 4, 6, &c. & eligendo n Sinus diversos erit.

Lib. I.

$$\sin. 2z = 2\sin. z. \sin. \left(\frac{\pi}{2} - z\right)$$

$$\sin. 4z = 8\sin. z. \sin. \left(\frac{\pi}{4} - z\right). \sin. \left(\frac{\pi}{4} + z\right). \sin. \left(\frac{\pi}{2} - z\right)$$

$$\sin. 6z = 32\sin. z. \sin. \left(\frac{\pi}{6} - z\right). \sin. \left(\frac{\pi}{6} + z\right). \sin. \left(\frac{2\pi}{6} - z\right) \times$$
$$\sin. \left(\frac{2\pi}{6} + z\right). \sin. \left(\frac{3\pi}{6} - z\right)$$

$$\sin. 8z = 128\sin. z. \sin. \left(\frac{\pi}{8} - z\right). \sin. \left(\frac{\pi}{8} + z\right). \sin. \left(\frac{2\pi}{8} - z\right) \times$$
$$\sin. \left(\frac{2\pi}{8} + z\right). \sin. \left(\frac{3\pi}{8} - z\right). \sin. \left(\frac{3\pi}{8} + z\right). \sin. \left(\frac{4\varpi}{8} - z\right)$$

240. Patet ergo fore generatim

$$\sin. nz = 2^{n-1} \sin. z. \sin. \left(\frac{\varpi}{n} - z\right). \sin. \left(\frac{\pi}{n} + z\right). \sin. \left(\frac{2\pi}{n} - z\right) \times$$
$$\sin. \left(\frac{2\pi}{n} + z\right). \sin. \left(\frac{3\pi}{n} - z\right). \sin. \left(\frac{3\pi}{n} + z\right) \ldots \sin. \left(\frac{1}{2}\pi - z\right)$$

si n fuerit numerus par. Quod si autem hæc cum superiori, ubi n erat numerus impar, comparetur tanta similitudo adesse deprehenditur, ut utramque in unam redigere liceat. Erit ergo, sive n fuerit numerus par sive impar,

$$\sin. nz = 2^{n-1} \sin. z. \sin. \left(\frac{\pi}{n} - z\right). \sin. \left(\frac{\varpi}{n} + z\right). \sin. \left(\frac{2\pi}{n} - z\right) \times$$
$$\sin. \left(\frac{2\varpi}{n} + z\right). \sin. \left(\frac{3\pi}{n} - z\right). \sin. \left(\frac{3\pi}{n} + z\right) \text{ \&c.}$$

donec tot habeantur Factores, quot numerus n continet unitates.

241. Expressiones istæ, quibus Sinus Angulorum multiplorum per Factores exponuntur, non parum utilitatis afferre possunt ad Logarithmos Sinuum Angulorum multiplorum inveniendos, itemque ad plures expressiones Sinuum per Factores, quales supra (§. 184) dedimus, reperiendas. Erit autem

sin.

$$\text{ſin. } z = 1 \text{ ſin. } z$$

$$\text{ſin. } 2z = 2 \text{ ſin. } z . \text{ſin.} \left(\frac{\pi}{2} - z\right)$$

$$\text{ſin. } 3z = 4 \text{ ſin. } z . \text{ſin.} \left(\frac{\pi}{3} - z\right) . \text{ſin.} \left(\frac{\pi}{3} + z\right)$$

$$\text{ſin. } 4z = 8 \text{ ſin. } z . \text{ſin.} \left(\frac{\pi}{4} - z\right) . \text{ſin.} \left(\frac{\pi}{4} + z\right) . \text{ſin.} \left(\frac{2\pi}{4} - z\right)$$

$$\text{ſin. } 5z = 16 \text{ ſin. } z . \text{ſin.} \left(\frac{\pi}{5} - z\right) . \text{ſin.} \left(\frac{\pi}{5} + z\right) . \text{ſin.} \left(\frac{2\pi}{5} - z\right) \times \text{ſin.} \left(\frac{2\pi}{5} + z\right)$$

$$\text{ſin. } 6z = 32 \text{ ſin. } z . \text{ſin.} \left(\frac{\pi}{6} - z\right) . \text{ſin.} \left(\frac{\pi}{6} + z\right) . \text{ſin.} \left(\frac{2\pi}{6} - z\right) \times \text{ſin.} \left(\frac{2\pi}{6} + z\right) . \text{ſin.} \left(\frac{3\pi}{6} - z\right)$$

&c.

242. Cum deinde ſit $\frac{\text{ſin. } 2nz}{\text{ſin. } nz} = 2\text{coſ. } nz$, Coſinus Angulorum multiplorum ſimili modo per Factores exprimentur.

$$\text{coſ. } z = 1 \text{ ſin.} \left(\frac{\varpi}{2} - z\right)$$

$$\text{coſ. } 2z = 2 \text{ ſin.} \left(\frac{\varpi}{4} - z\right) . \text{ſin.} \left(\frac{\pi}{4} + z\right)$$

$$\text{coſ. } 3z = 4 \text{ ſin.} \left(\frac{\varpi}{6} - z\right) . \text{ſin.} \left(\frac{\pi}{6} + z\right) . \text{ſin.} \left(\frac{3\pi}{6} - z\right)$$

$$\text{coſ. } 4z = 8 \text{ ſin.} \left(\frac{\varpi}{8} - z\right) . \text{ſin.} \left(\frac{\pi}{8} + z\right) . \text{ſin.} \left(\frac{3\pi}{8} - z\right) \times \text{ſin.} \left(\frac{3\varpi}{8} + z\right)$$

$$\text{coſ. } 5z = 16 \text{ ſin.} \left(\frac{\varpi}{10} - z\right) . \text{ſin.} \left(\frac{\varpi}{10} + z\right) . \text{ſin.} \left(\frac{3\pi}{10} - z\right) \times \text{ſin.} \left(\frac{3\varpi}{10} + z\right) . \text{ſin.} \left(\frac{5\pi}{10} - z\right)$$

& generaliter

LIB. I.

$$cos. nz = 2^{n-1} \, sin.\left(\frac{\pi}{2n} - z\right). sin.\left(\frac{\pi}{2n} + z\right). sin.\left(\frac{3\pi}{2n} - z\right) \times$$
$$sin.\left(\frac{3\pi}{2n} + z\right). sin.\left(\frac{5\pi}{2n} - z\right) \text{ \&c.},$$

quoad tot habeantur Factores quot numerus n continet unitates.

243. Eædem expressiones prodibunt ex consideratione Cosinuum Arcuum multiplorum, si enim fuerit $cos. z = y$, erit ut sequitur

$$
\begin{array}{lllll}
cos. 0z = & 1 & & & \\
cos. 1z = & y & & & \\
cos. 2z = & 2yy & - 1 & & \\
cos. 3z = & 4y^3 & - 3y & & \\
cos. 4z = & 8y^4 & - 8yy & + 1 & \\
cos. 5z = & 16y^5 & - 20y^3 & + 5y & \\
cos. 6z = & 32y^6 & - 48y^4 & + 18yy & - 1 \\
cos. 7z = & 64y^7 & - 112y^5 & + 56y^3 & - 7y
\end{array}
$$

& generaliter.

$$cos. nz = 2^{n-1} y^n - \frac{n}{1} 2^{n-3} y^{n-2} + \frac{n(n-3)}{1.\ 2} 2^{n-5} y^{n-4} - \frac{n(n-4)(n-5)}{1.\ 2.\ 3} 2^{n-7} y^{n-6} + \frac{n(n-5)(n-6)(n-7)}{1.\ 2.\ 3.\ 4} 2^{n-9} y^{n-8} - \text{\&c.},$$

cujus æquationis, cum sit $cos. nz = cos. (2\pi - nz) = cos. (2\pi + nz) = cos. (4\pi \pm nz) = cos. (6\pi \pm nz)$ &c., erunt radices ipsius y hæ: $cos. z$; $cos. \left(\frac{2\pi}{n} \pm z\right)$; $cos. \left(\frac{4\pi}{n} \pm z\right)$; $cos. \left(\frac{6\pi}{n} \pm z\right)$ &c., quarum formularum tot diversæ sunt pro y eligendæ quot dantur; dantur autem tot, quot n continet unitates.

244. Pri-

244. Primum igitur patet, ob terminum ſecundum deficientem excepto caſu $n = 1$, fore ſummam harum radicum omnium $= 0$. Erit ergo

$$0 = cof.z + cof.\left(\frac{2\varpi}{n} - z\right) + cof.\left(\frac{2\varpi}{n} + z\right) + cof.\left(\frac{4\varpi}{n} - z\right) + cof.\left(\frac{4\varpi}{n} + z\right) + \&c.,$$

ſumendo tot terminos quot n continet unitates: Hæc autem æqualitas ſponte ſe offert ſi n ſit numerus par, cum quivis terminus ab alio ſui negativo deſtruatur. Contemplemur ergo numeros impares, unitate excluſa, eritque, ob $cof.v = - cof.(\pi - v)$

$$0 = cof.z - cof.\left(\frac{\pi}{3} - z\right) - cof.\left(\frac{\pi}{3} + z\right)$$

$$0 = cof.z - cof.\left(\frac{\pi}{5} - z\right) - cof.\left(\frac{\pi}{5} + z\right) + cof.\left(\frac{2\pi}{5} - z\right) + cof.\left(\frac{2\pi}{5} + z\right)$$

$$0 = cof.z - cof.\left(\frac{\pi}{7} - z\right) - cof.\left(\frac{\pi}{7} + z\right) + cof.\left(\frac{2\pi}{7} - z\right) + cof.\left(\frac{2\pi}{7} + z\right) - cof.\left(\frac{3\pi}{7} - z\right) - cof.\left(\frac{3\pi}{7} + z\right)$$

& generaliter, ſi fuerit n numerus impar quicunque, erit

$$0 = cof.z - cof.\left(\frac{\pi}{n} - z\right) - cof.\left(\frac{\pi}{n} + z\right) + cof.\left(\frac{2\pi}{n} - z\right) + cof.\left(\frac{2\pi}{n} + z\right) - cof.\left(\frac{3\pi}{n} - z\right) - cof.\left(\frac{3\pi}{n} + z\right) + cof.\left(\frac{4\pi}{n} - z\right) + cof.\left(\frac{4\pi}{n} + z\right) - \&c.,$$

ſumendo tot terminos, quot numerus n continet unitates: oportet autem n eſſe numerum imparem unitate majorem, uti jam monuimus.

245. Quod

LIB. I.

245. Quod ad productum ex omnibus attinet, variæ quidem prodeunt expressiones, prout n fuerit numerus vel impar, vel impariter par, vel pariter par: omnes autem comprehenduntur in expressione generali (§. 242.) inventa, si singuli Sinus in Cosinus transmutentur: Erit scilicet

$$cos.\ z = 1\ cos.\ z$$

$$cos.\ 2z = 2\ cos.\left(\frac{\pi}{4} + z\right).\ cos.\left(\frac{\pi}{4} - z\right)$$

$$cos.\ 3z = 4\ cos.\left(\frac{2\pi}{6} + z\right).\ cos.\left(\frac{2\pi}{6} - z\right).\ cos.\ z$$

$$cos.\ 4z = 8\ cos.\left(\frac{3\pi}{8} + z\right).\ cos.\left(\frac{3\pi}{8} - z\right).\ cos.\left(\frac{\pi}{8} + z\right) \times cos.\left(\frac{\pi}{8} - z\right)$$

$$cos.\ 5z = 16\ cos.\left(\frac{4\pi}{8} + z\right).\ cos.\left(\frac{4\pi}{8} - z\right).\ cos.\left(\frac{2\pi}{8} + z\right) \times cos.\left(\frac{2\pi}{8} - z\right)\ cos.\ z$$

& generaliter

$$cos.\ nz = 2^{n-1}\ cos.\left(\frac{n-1}{n}\pi + z\right).\ cos.\left(\frac{n-1}{n}\varpi - z\right) \times$$
$$cos.\left(\frac{n-3}{n}\pi + z\right).\ cos.\left(\frac{n-3}{n}\varpi - z\right) \times$$
$$cos.\left(\frac{n-5}{n}\varpi + z\right).\ cos.\left(\frac{n-5}{n}\varpi - z\right) \times$$
$$cos.\left(\frac{n-7}{n}\varpi + z\right)\ \&c.,$$

sumtis tot Factoribus, quot numerus n continet unitates.

246. Sit n numerus impar, atque æquatio incipiatur ab unitate, erit

$0 = 1 \mp \frac{ny}{cos.\ nz} + \&c.$, ubi signum superius valet si n fuerit numerus impar formæ $4m + 1$, inferius si $n = 4m - 1$. Hinc erit

+

$$+\frac{1}{cos.z}=\frac{1}{cos.z}$$

$$-\frac{3}{cos.3z}=\frac{1}{cos.z}-\frac{1}{cos.(\frac{\pi}{3}-z)}-\frac{1}{cos.(\frac{\pi}{3}+z)}$$

$$+\frac{5}{cos.5z}=\frac{1}{cos.z}-\frac{1}{cos.(\frac{\pi}{5}-z)}-\frac{1}{cos.(\frac{\varpi}{5}+z)}+\frac{1}{cos.(\frac{2\varpi}{5}-z)}+\frac{1}{cos.(\frac{2\varpi}{5}+z)}$$

& generaliter, posito $n = 2m + 1$, erit

$$\frac{n}{cos.nz}=\frac{2m+1}{cos.(2m+1)z}=\frac{1}{cos.(\frac{m}{n}\pi+z)}+\frac{1}{cos.(\frac{m}{n}\pi-z)}-\frac{1}{cos.(\frac{m-1}{n}\pi+z)}-\frac{1}{cos.(\frac{m-1}{n}\pi-z)}+\frac{1}{cos.(\frac{m-2}{n}\pi+z)}+\frac{1}{cos.(\frac{m-2}{n}\pi-z)}-\frac{1}{cos.(\frac{m-3}{n}\varpi+z)}-\&c.$$

sumendis tot terminis, quot n continet unitates.

247. Cum ergo sit $\frac{1}{cos.v}=sec.v.$, hinc pro Secantibus insignes proprietates deducuntur, erit nempe

$$sec.\ z = sec.z.$$

$$3sec.3z = sec.(\frac{\varpi}{3}+z)+sec.(\frac{\pi}{3}-z)-sec.(\frac{0\pi}{3}+z)$$

$$5sec.5z = sec.(\frac{2\pi}{5}+z)+sec.(\frac{2\pi}{5}-z)-sec.(\frac{\pi}{5}+z)-sec.(\frac{\pi}{5}-z)+sec.(\frac{0\pi}{5}+z)$$

$$7 sec. 7z = sec.(\frac{3\pi}{7}+z) + sec.(\frac{3\pi}{7}-z) - sec.(\frac{2\pi}{7}+z) - sec.(\frac{2\pi}{7}-z) + sec.(\frac{\varpi}{7}+z) + sec.(\frac{\varpi}{7}-z) - sec.(\frac{0\varpi}{7}+z)$$

& generaliter, posito $n = 2m + 1$, erit

$$n. sec. nz = sec.(\frac{m}{n}\pi+z) + sec.(\frac{m}{n}\varpi-z) - sec.(\frac{m-1}{n}\pi+z) - sec.(\frac{m-1}{n}\varpi-z) + sec.(\frac{m-2}{n}\pi+z) + sec.(\frac{m-2}{n}\varpi-z) - sec.(\frac{m-3}{n}\pi+z) - sec.(\frac{m-3}{n}\varpi-z) + sec.(\frac{m-4}{n}\pi+z) + \ldots\ldots\ldots\ldots \pm sec. z.$$

248. Pro Cosecantibus autem erit ex §. 237.

$$cosec. z = cosec. z$$

$$3 cosec. 3z = cosec. z + cosec.(\frac{\pi}{3}-z) - cosec.(\frac{\pi}{3}+z)$$

$$5 cosec. 5z = cosec. z + cosec.(\frac{\pi}{5}-z) - cosec.(\frac{\pi}{5}+z) - cosec.(\frac{2\pi}{5}-z) + cosec.(\frac{2\pi}{5}+z)$$

$$7 cosec. 7z = cosec. z + cosec.(\frac{\pi}{7}-z) - cosec.(\frac{\pi}{7}+z) - cosec.(\frac{2\pi}{7}-z) + cosec.(\frac{2\pi}{7}+z) + cosec.(\frac{3\pi}{7}-z) - cosec.(\frac{3\pi}{7}+z)$$

& generaliter, ponendo $n = 2m + 1$, erit

$$n.\,cosec.\,nz = cosec.\,z + cosec.(\frac{\pi}{n} - z) - cosec.(\frac{\pi}{n} + z) - cosec.(\frac{2\pi}{n} - z) + cosec.(\frac{2\pi}{n} + z) + cosec.(\frac{3\pi}{n} - z) - cosec.(\frac{3\pi}{n} + z) - \ldots\ldots\ldots\ldots \mp cosec.(\frac{m\pi}{n} - z) \pm cosec.(\frac{m\pi}{n} + z)$$

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ubi signa superiora valent si m fuerit numerus par, inferiora si m sit impar.

249. Cum sit, uti supra vidimus, $cos.\,nz \pm \sqrt{-1}.\,sin.\,nz = (cos.\,z \pm \sqrt{-1}.\,sin.\,z)^n$, erit $cos.\,nz = \frac{(cos.\,z + \sqrt{-1}.\,sin.\,z)^n + (cos.\,z - \sqrt{-1}.\,sin.\,z)^n}{2}$, & $sin.\,nz = \frac{(cos.\,z + \sqrt{-1}.\,sin.\,z)^n - (cos.\,z - \sqrt{-1}.\,sin.\,z)^n}{2\sqrt{-1}}$, erit

$$tang.\,nz = \frac{(cos.\,z + \sqrt{-1}.\,sin.\,z)^n - (cos.\,z - \sqrt{-1}.\,sin.\,z)^n}{(cos.\,z + \sqrt{-1}.\,sin.\,z)^n\sqrt{-1} + (cos.\,z - \sqrt{-1}.\,sin.\,z)^n\sqrt{-1}}.$$

Ponamus $tang.\,z = \frac{sin.\,z}{cos.\,z} = t$, erit $tang.\,nz = \frac{(1 + t\sqrt{-1})^n - (1 - t\sqrt{-1})^n}{(1 + t\sqrt{-1})^n\sqrt{-1} + (1 - t\sqrt{-1})^n\sqrt{-1}}$, unde oriuntur Tangentes Angulorum multiplorum sequentes

$$tang.\,z = t$$

$$tang.\,2z = \frac{2t}{1 - tt}$$

$$tang.\,3z = \frac{3t - t^3}{1 - 3tt}$$

$$tang.\,4z = \frac{4t - 4t^3}{1 - 6tt + t^4}$$

$$tang.\,5z = \frac{5t - 10t^3 + t^5}{1 - 10tt + 5t^4}$$

& generaliter

$$tang.nz = \frac{nt - \frac{n(n-1)(n-2)}{1.\ 2.\ 3}t^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3\ 4\ 5}t^5 - \&c.}{1 - \frac{n(n-1)}{1.\ 2}tt + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}t^4 - \&c.}.$$

Cum jam ſit $tang.\ nz = tang.(\varpi + nz) = tang.(2\pi + nz) = tang.(3\varpi + nz)$ &c., erunt valores ipſius t, ſeu radices æquationis, hæ, $tang.z$; $tang.(\frac{\pi}{n} + z)$; $tang.(\frac{2\pi}{n} + z)$; $tang.(\frac{3\pi}{n} + z)$; &c., quarum numerus eſt n.

250. Quod ſi æquatio ab unitate incipiat, erit

$$0 = 1 - \frac{nt}{tang.\ nz} - \frac{n(n-1)tt}{1.\ 2} + \frac{n(n-1)(n-2)t^3}{1.\ 2.\ 3\ tang.\ nz} + \&c..$$

Ex comparatione ergo coëfficientium cum radicibus, erit

$$n.cot.nz = cot.z + cot.(\frac{\pi}{n} + z) + cot.(\frac{2\pi}{n} + z) + cot.(\frac{3\pi}{n} + z) + cot.(\frac{4\varpi}{n} + z) + \dots\dots + cot.(\frac{n-1}{n}\pi + z)$$

deinde erit ſumma quadratorum harum Cotangentium omnium $= \frac{nn}{(sin.\ nz)^2} - n$, ſimilique modo ulteriores Poteſtates poſſunt definiri. Ponendo autem loco n numeros definitos, erit.

$$cot.\ z = cot.z$$

$$2cot.2z = cot.z + cot.(\frac{\pi}{2} + z)$$

$$3cot.3z = cot.z + cot.(\frac{\pi}{3} + z) + cot.(\frac{2\pi}{3} + z)$$

$$4cot.4z = cot.z + cot.(\frac{\pi}{4} + z) + cot.(\frac{2\pi}{4} + z) + cot.(\frac{3\pi}{4} + z)$$

$$5\cot. 5z = \cot. z + \cot.\left(\frac{\pi}{5}+z\right) + \cot.\left(\frac{2\pi}{5}+z\right) + \cot.\left(\frac{3\pi}{5}+z\right) + \cot.\left(\frac{4\pi}{5}+z\right).$$

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251. Quia vero est $\cot. v = -\cot.(\pi - v)$, erit

$$\cot. z = \cot. z$$

$$2\cot. 2z = \cot. z - \cot.\left(\frac{\pi}{2}-z\right)$$

$$3\cot. 3z = \cot. z - \cot.\left(\frac{\pi}{3}-z\right) + \cot.\left(\frac{\pi}{3}+z\right)$$

$$4\cot. 4z = \cot. z - \cot.\left(\frac{\pi}{4}-z\right) + \cot.\left(\frac{\pi}{4}+z\right) - \cot.\left(\frac{2\pi}{4}-z\right)$$

$$5\cot. 5z = \cot. z - \cot.\left(\frac{\pi}{5}-z\right) + \cot.\left(\frac{\pi}{5}+z\right) - \cot.\left(\frac{2\pi}{5}-z\right) + \cot.\left(\frac{2\pi}{5}+z\right)$$

& generaliter

$$n.\cot. nz = \cot. z - \cot.\left(\frac{\pi}{n}-z\right) + \cot.\left(\frac{\pi}{n}+z\right) - \cot.\left(\frac{2\pi}{n}-z\right) + \cot.\left(\frac{2\pi}{n}+z\right) - \cot.\left(\frac{3\pi}{n}-z\right) + \cot.\left(\frac{3\pi}{n}+z\right) - \&c.$$

donec tot habeantur termini, quot numerus n continet unitates.

252. Incipiamus æquationem inventam a Potestate summa, ubi primum distingendi sunt casus, quibus n est vel numerus par, vel impar. Sit n numerus impar, $n = 2m + 1$ erit

LIB. I. $t - tang.\,z = 0$

$t^3 - 3tt.tang.\,3z - 3t + tang.\,3z = 0$

$t^5 - 5t^4.tang.\,5z - 10t^3 + 10tt.tang.\,5z + 5t - tang.\,5z = 0$

& generaliter

$$t^n - nt^{n-1}\, tang.\, nz - \dots\dots\dots\dots \mp tang.\, nz = 0$$

ubi ſignum ſuperius — valet, ſi m ſit numerus par, inferius + ſi m ſit numerus impar. Erit ergo ex coëfficiente ſecundi termini

$$tang.\, z = tang.\, z$$

$$3tang.\, 3z = tang.\, z + tang.\,(\frac{\pi}{3} + z) + tang.\,(\frac{2\pi}{3} + z)$$

$$5tang.\, 5z = tang.\, z + tang.\,(\frac{\varpi}{5} + z) + tang.\,(\frac{2\varpi}{5} + z) + tang.\,(\frac{3\varpi}{5} + z) + tang.\,(\frac{4\varpi}{5} + z).$$

&c.

253. Cum igitur ſit $tang.\, v = - tang.\,(\pi - v)$, Anguli recto majores ad Angulos recto minores reducuntur, eritque

$$tang.\, z = tang.\, z$$

$$3tang.\, 3z = tang.\, z - tang.\,(\frac{\varpi}{3} - z) + tang.\,(\frac{\pi}{3} + z)$$

$$5tang.\, 5z = tang.\, z - tang.\,(\frac{\varpi}{5} - z) + tang.\,(\frac{\varpi}{5} + z) - tang.\,(\frac{2\pi}{5} - z) + tang.\,(\frac{2\pi}{5} + z)$$

$$7tang.\, 7z = tang.\, z - tang.\,(\frac{\pi}{7} - z) + tang.\,(\frac{\pi}{7} + z) - tang.\,(\frac{2\pi}{7} - z) + tang.\,(\frac{2\pi}{7} + z) - tang.\,(\frac{3\pi}{7} - z) + tang.\,(\frac{3\varpi}{7} + z)$$

& gene-

& generaliter, si $n = 2m + 1$, erit

$$n.tang.nz = tang.z - tang.(\frac{\pi}{n} - z) + tang.(\frac{\pi}{n} + z) - tang.(\frac{2\pi}{n} - z) + tang.(\frac{2\varpi}{n} + z) - tang.(\frac{3\varpi}{n} - z) + \ldots\ldots\ldots\ldots - tang.(\frac{m\pi}{n} - z) + tang.(\frac{m\pi}{n} + z).$$

254. Tum vero productum ex his Tangentibus omnibus erit $= tang.\, nz$, propterea quod per signorum negativorum numerum alternatim parem & imparem, superior signorum ambiguitas tollitur. Sic erit

$$tang.\ z = tang.z$$

$$tang.3z = tang.z.\, tang.(\frac{\pi}{3} - z).\, tang.(\frac{\varpi}{3} + z)$$

$$tang.5z = tang.z.\, tang.(\frac{\pi}{5} - z).\, tang.(\frac{\pi}{5} + z).tang.(\frac{2\pi}{5} - z) \times tang.(\frac{2\pi}{5} + z)$$

& generaliter, si $n = 2m + 1$, erit

$$tang.nz = tang.z.\, tang.(\frac{\pi}{n} - z).tang.(\frac{\pi}{n} + z).tang.(\frac{2\pi}{n} - z) \times tang.(\frac{2\pi}{n} + z).tang.(\frac{3\pi}{n} - z)\ldots\ldots\ldots \times tang.(\frac{m\pi}{n} - z).tang.(\frac{m\pi}{n} + z).$$

255. Sit jam n numerus par, atque, incipiendo a Potestate summa, erit

$$tt + 2t.\, cot.\, 2z - 1 = 0$$

$$t^4 + 4t^3.cot.\, 4z - 6tt - 4t.cot.\, 4z + 1 = 0$$

&

& generaliter, si $n = 2m$, erit

$$t^n + nt^{n-1} \, cot. \, nz - \ldots\ldots\ldots\ldots\ldots\ldots \mp 1 = 0$$

ubi signum superius $-$ valet si m sit numerus impar, inferius $+$ si m sit par. Comparando ergo radices cum coëfficiente secundi termini, erit

$$-2 \, cot. \, 2z = tang. \, z + tang. \left(\frac{\pi}{2} + z\right)$$

$$-4 \, cot. \, 4z = tang. \, z + tang. \left(\frac{\pi}{4} + z\right) + tang. \left(\frac{2\pi}{4} + z\right) + tang. \left(\frac{3\pi}{4} + z\right)$$

$$-6 \, cot. \, 6z = tang. \, z + tang. \left(\frac{\pi}{6} + z\right) + tang. \left(\frac{2\pi}{6} + z\right) + tang. \left(\frac{3\pi}{6} + z\right) + tang. \left(\frac{4\pi}{6} + z\right) + tang. \left(\frac{5\pi}{6} + z\right).$$

&c.

256. Cum sit $tang. \, v = - tang. \, (\pi - v)$, sequentes formabuntur æquationes

$$2 \, cot. \, 2z = - tang. \, z + tang. \left(\frac{\pi}{2} - z\right)$$

$$4 \, cot. \, 4z = - tang. \, z + tang. \left(\frac{\pi}{4} - z\right) - tang. \left(\frac{\pi}{4} + z\right) + tang. \left(\frac{2\pi}{4} - z\right)$$

$$6 \, cot. \, 6z = - tang. \, z + tang. \left(\frac{\pi}{6} - z\right) - tang. \left(\frac{\pi}{6} + z\right) + tang. \left(\frac{2\pi}{6} - z\right) - tang. \left(\frac{2\pi}{6} + z\right) + tang. \left(\frac{3\pi}{6} - z\right)$$

&

& generaliter, fi $n = 2m$, erit

$$n.\cot. nz = -\,tang.z + tang.\left(\frac{\varpi}{n} - z\right) - tang.\left(\frac{\varpi}{n} + z\right) + tang.\left(\frac{2\varpi}{n} - z\right) - tang.\left(\frac{2\varpi}{n} + z\right) + tang.\left(\frac{3\varpi}{n} - z\right) - tang.\left(\frac{3\varpi}{n} + z\right) + \ldots\ldots\ldots\ldots + tang.\left(\frac{m\varpi}{n} - z\right).$$

257. Per has formas iterum ambiguitas producti ex omnibus radicibus deftruitur; eritque idcirco

$$1 = tang.z.\,tang.\left(\frac{\varpi}{2} - z\right)$$

$$1 = tang.z.\,tang.\left(\frac{\varpi}{4} - z\right).tang.\left(\frac{\varpi}{4} + z\right).tang.\left(\frac{2\varpi}{4} - z\right)$$

$$1 = tang.z.\,tang.\left(\frac{\varpi}{6} - z\right).tang.\left(\frac{\varpi}{6} + z\right).tang.\left(\frac{2\varpi}{6} - z\right) \times tang.\left(\frac{2\varpi}{6} + z\right).tang.\left(\frac{3\varpi}{6} - z\right).$$

&c.

Harum vero æquationum ratio ftatim fponte in oculos incurrit, cum perpetuo bini Anguli reperiantur, quorum alter eft alterius complementum ad rectum. Hujufmodi ergo binorum Angulorum Tangentes productum dant $= 1$; ideoque omnium productum unitati debet effe æquale.

258. Quoniam Sinus & Cofinus Angulorum progreffionem arithmeticam conftituentium Seriem recurrentem præbent, per Caput præcedens fumma hujufmodi Sinuum & Cofinuum quotcunque exhiberi poterit. Sint Anguli in arithmetica progreffione

$$a,\ a+b,\ a+2b,\ a+3b,\ a+4b,\ a+5b,\ \&c.$$

& quæratur primo fumma Sinuum horum Angulorum in infinitum progredientium; ponatur ergo

LIB. I. $s = \sin. a + \sin. (a+b) + \sin. (a+2b) + \sin. (a+3b) + \&c.$

& quia hæc Series est recurrens, cujus ſcala relationis eſt $2\cos. b, -1$, orietur hæc Series ex evolutione fractionis, cujus denominator eſt $1 - 2z.\cos. b + zz$, poſito $z = 1$. Ipſa vero fractio erit $= \frac{\sin. a + z(\sin.(a+b) - 2\sin. a.\cos. b)}{1 - 2z.\cos. b + zz}$, quare, facto $z = 1$, erit $s = \frac{\sin. a + \sin.(a+b) - 2\sin. a.\cos. b}{2 - 2\cos. b} = \frac{\sin. a - \sin.(a-b)}{2(1-\cos. b)}$, ob $2\sin. a.\cos. b = \sin.(a+b) + \sin.(a-b)$. Cum autem ſit $\sin. f - \sin. g = 2\cos. \frac{f+g}{2}.\sin. \frac{f-g}{2}$, erit $\sin. a - \sin.(a-b) = 2\cos.(a - \frac{1}{2}b).\sin. \frac{1}{2}b$: & $1 - \cos. b = 2(\sin. \frac{1}{2}b)^2$, unde erit $s = \frac{\cos.(a-\frac{1}{2}b)}{2\sin. \frac{1}{2}b}$.

259. Hinc itaque ſumma quotcunque Sinuum, quorum Arcus in arithmetica progreſſione incedunt, aſſignari poterit; quæratur nempe ſumma hujus progreſſionis

$$\sin. a + \sin.(a+b) + \sin.(a+2b) + \sin.(a+3b) + \ldots\ldots + \sin.(a+nb).$$

Quia ſumma hujus progreſſionis in infinitum continuatæ eſt $\frac{\cos.(a-\frac{1}{2}b)}{2\sin. \frac{1}{2}b}$, conſiderentur termini ultimum ſequentes in infinitum hi

$$\sin.(a+(n+1)b) + \sin.(a+(n+2)b) + \sin.(a+(n+3)b) + \&c.$$

quia horum Sinuum ſumma eſt $= \frac{\cos.(a+(n+\frac{1}{2})b)}{2\sin. \frac{1}{2}b}$; ſi hæc a priori ſubtrahatur, remanebit ſumma quæſita. Scilicet, ſi fuerit

$$s = \sin. a + \sin.(a+b) + \sin.(a+2b) + \ldots\ldots + \sin.(a+nb),$$

erit $s = \frac{\cos.(a-\frac{1}{2}b) - \cos.(a+(n+\frac{1}{2})b)}{2\sin. \frac{1}{2}b} = \frac{\sin.(a+\frac{1}{2}nb)\sin.\frac{1}{2}(n+1)b}{\sin. \frac{1}{2}b}$

260. Pari

260. Pari modo, ſi conſideretur ſumma Coſinuum, atque ponatur Cap. XIV.

$$s = cos.\, a + cos.\,(a+b) + cos.\,(a+2b) + cos.\,(a+3b) + \&c.$$

in infinitum, erit $s = \frac{cos.\, a + z(cos.\,(a+b) - 2cos.a.cos.b)}{1 - 2z.\, cos.\, b + zz}$, poſito $z = 1$. Quare, ob $2cos.a.cos.b = cos.(a-b) + cos.(a+b)$, fiet $s = \frac{cos.a - cos.(a-b)}{2(1 - cos.\, b)}$. At eſt $cos.f - cos.g = 2\, fin.\frac{f+g}{2} \times fin.\, \frac{g-f}{2}$; unde erit $cos.a - cos.(a-b) = -2fin.(a - \frac{1}{2}b) \times fin.\, \frac{1}{2}b$, & ob $1 - cos.\, b = 2(fin.\, \frac{1}{2}b)^2$, erit $s = -\frac{fin.\,(a - \frac{1}{2}b)}{2\,fin.\,\frac{1}{2}b}$. Quare, cum ſimili modo ſit hujus Seriei

$$cos.(a + (n+1)b) + cos.(a + (n+2)b) + cos.(a + (n+3)b) + \&c.$$

ſumma ſit $= -\frac{fin.(a + (n+\frac{1}{2})b)}{2\,fin.\frac{1}{2}b}$, ſi hæc ab illa ſubtrahatur, relinquetur ſumma hujus Seriei

$$s = cos.a + cos.(a+b) + cos.(a+2b) + cos.(a+3b) + \ldots\ldots\ldots + cos.(a+nb):$$

eritque $s = -\frac{fin.(a - \frac{1}{2}b) + fin.\,(a + (n+\frac{1}{2})b)}{2\,fin.\,\frac{1}{2}b} = \frac{cos.(a + \frac{1}{2}nb).fin.\frac{1}{2}(n+1)b}{fin.\,\frac{1}{2}b}$.

261. Plurimæ aliæ quæſtiones circa Sinus & Tangentes ex principiis allatis reſolvi poſſent; cujuſmodi ſunt, ſi quadrata, altioreſve Poteſtates Sinuum, Tangentiumve ſummari deberent, verum quia hæc ex reliquis æquationum ſuperiorum coëfficientibus ſimiliter derivantur, iis hic diutius non immoror. Quod autem ad has poſtremas ſummationes attinet, notandum eſt quamcunque Sinuum Coſinuumque Poteſtatem per ſingulos Sinus Coſinusve explicari poſſe, quod, ut clarius perſpiciatur, breviter exponamus.

LIB. I. 261. Ad hoc expediendum juvabit ex præcedentibus hæc Lemmata depromsisse

$$2\sin.a.\sin.z = \cos.(a - z) - \cos.(a + z)$$
$$2\cos.a.\sin.z = \sin.(a + z) - \sin.(a - z)$$
$$2\sin.a\cos.z = \sin.(a + z) + \sin.(a - z)$$
$$2\cos.a.\cos.z = \cos.(a - z) + \cos.(a + z)$$

Hinc igitur primum Potestates Sinuum reperiuntur

$$\sin.z = \sin.z$$
$$2(\sin.z)^2 = 1 - \cos.2z$$
$$4(\sin.z)^3 = 3\sin.z - \sin.3z$$
$$8(\sin.z)^4 = 3 - 4\cos.2z + \cos.4z$$
$$16(\sin.z)^5 = 10\sin.z - 5\sin.3z + \sin.5z$$
$$32(\sin.z)^6 = 10 - 15\cos.2z + 6\cos.4z - \cos.6z$$
$$64(\sin.z)^7 = 35\sin.z - 21\sin.3z + 7\sin.5z - \sin.7z$$
$$128(\sin.z)^8 = 35 - 56\cos.2z + 28\cos.4z - 8\cos.6z + \cos.8z$$
$$256(\sin.z)^9 = 126\sin.z - 84\sin.3z + 36\sin.5z - 9\sin.7z + \sin.9z$$

&c.

Lex, qua hi coëfficientes progrediuntur, ex unciis Binomii elevati intelligitur, nisi quod numerus absolutus in Potestatibus paribus semissis tantum sit ejus, quem unciæ præbent.

263. Pari modo Potestates Cosinuum definientur

$$\cos.z = \cos.z$$
$$2(\cos.z)^2 = 1 + \cos.2z$$
$$4(\cos.z)^3 = 3\cos.z + \cos.3z$$
$$8(\cos.z)^4 = 3 + 4\cos.2z + \cos.4z$$
$$16(\cos.z)^5 = 10\cos.z + 5\cos.3z + \cos.5z$$
$$32(\cos.z)^6 = 10 + 15\cos.2z + 6\cos.4z + \cos.6z$$
$$64(\cos.z)^7 = 35\cos.z + 21\cos.3z + 7\cos.5z + \cos.7z$$

&c.

Hic ratione legis progressionis eadem sunt monenda quæ circa Sinus notavimus.

CAPUT

CAPUT XV.

De Seriebus ex evolutione Factorum ortis.

264. Sit propositum productum ex Factoribus, numero sive finitis sive infinitis, constans hujusmodi

$$(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)(1+\varepsilon z)(1+\zeta z)\ \&c.,$$

quod, si per multiplicationem actualem evolvatur, det

$$1+Az+Bz^2+Cz^3+Dz^4+Ez^5+Fz^6+\&c.,$$

atque manifestum est coëfficientes A, B, C, D, E, &c., ita formari ex numeris α, β, γ, δ, ε, ζ, &c., ut sit

$A = \alpha+\beta+\gamma+\delta+\varepsilon+\zeta+$ &c. $=$ summæ singulorum
$B =$ summæ Factorum ex binis diversis
$C =$ summæ Factorum ex ternis diversis
$D =$ summæ Factorum ex quaternis diversis
$E =$ summæ Factorum ex quinis diversis
&c.

donec perveniatur ad productum ex omnibus.

265. Quod si ergo ponatur $z=1$, productum hoc

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)\ \&c.$$

æquabitur unitati cum Serie numerorum omnium, qui ex his α, β, γ, δ, ε, &c., vel sumendis singulis, vel duobus pluribusve diversis in se multiplicandis, nascuntur. Atque si idem numerus duobus pluribusve modis resultare queat, etiam idem bis pluriesve in hac numerorum Serie occurret.

266. Si ponatur $z=-1$, productum hoc

 (1 —

LIB. I.

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon)\ \&c.$$

æquabitur unitati cum Serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, &c. vel sumendis singulis, vel duobus pluribusve diversis in se multiplicandis, nascuntur; ut ante quidem, verum hoc discrimine, ut ii numeri, qui vel ex singulis, vel ternis, vel quinis, vel numero imparibus nascuntur, sint negativi, illi vero, qui vel ex binis, vel quaternis, vel senis vel numero paribus resultant, sint affirmativi.

267. Scribantur pro $\alpha, \beta, \gamma, \delta$, &c., numeri primi omnes 2, 3, 5, 7, 11, 13, &c., atque hoc productum

$$(1+2)(1+3)(1+5)(1+7)(1+11)(1+13)\ \&c. = P$$

æquabitur unitati, cum Serie omnium numerorum vel primorum ipsorum, vel ex primis diversis per multiplicationem ortorum. Erit ergo

$$P = 1+2+3+5+6+7+10+11+13+14+15+17+\ \&c.,$$

in qua Serie omnes occurrunt numeri naturales, exceptis Potestatibus, iisque qui per quamvis Potestatem sunt divisibiles. Desunt scilicet numeri 4, 8, 9, 12, 16, 18 &c., quoniam sunt vel Potestates, ut 4, 8, 9, 16, &c., vel per Potestates divisibiles ut 12, 18, &c.

268. Simili modo res se habebit, si pro $\alpha, \beta, \gamma, \delta$, &c.. Potestates quæcunque numerorum primorum substituantur. Scilicet si ponamus

$$P = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\ \&c.,$$

erit enim multiplicatione instituta:

$$P = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{10^n}+\frac{1}{11^n}+\ \&c.$$

in

in quibus fractionibus omnes occurrunt numeri præter illos qui vel ipſi ſunt Poteſtates, vel per Poteſtatem quampiam diviſibiles. Cum enim omnes numeri integri ſint vel primi vel ex primis per multiplicationem compoſiti, hic ii tantum numeri excludentur, in quorum formationem idem numerus primus bis vel pluries ingreditur. CAP. XV.

269. Si numeri $\alpha, \beta, \gamma, \delta$, &c., negative capiantur, ut ante (266.) fecimus, atque ponatur

$$P = (1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n})$$

&c., erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{15^n} - \text{\&c.},$$

ubi iterum, ut ante, omnes occurrunt numeri præter Poteſtates ac diviſibiles per Poteſtates. Verum ipſi numeri primi, & qui ex ternis, quinis, numerove imparibus conſtant, ſignum habent præfixum —, qui autem ex binis, vel quaternis, vel ſenis, vel numero paribus formantur, ſignum habent +. Sic in hac Serie occurret terminus $\frac{1}{30^n}$, quia eſt $30 = 2 \cdot 3 \cdot 5$, neque adeo Poteſtatem complectitur, habebit vero hic terminus $\frac{1}{30^n}$ ſignum —, quia 30 eſt productum ex tribus numeris primis.

270. Conſideremus jam hanc expreſſionem

$$\frac{1}{(1 - \alpha z)(1 - \beta z)(1 - \gamma z)(1 - \delta z)(1 - \varepsilon z)\ \text{\&c.}}$$

quæ per diviſionem actualem evoluta præbeat hanc Seriem:

1 +

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \&c.$$

atque manifeſtum eſt coëfficientes A, B, C, D, E, &c., ſequenti modo ex numeris α, β, γ, δ, ε, &c., componi, ut ſit

A = ſummæ ſingulorum
B = ſummæ Factorum ex binis
C = ſummæ Factorum ex ternis
D = ſummæ Factorum ex quaternis
&c.
} non excluſis Factoribus iiſdem.

271. Poſito ergo $z = 1$, iſta expreſſio

$$\frac{1}{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon)\ \&c.}$$

æquabitur unitati cum Serie numerorum omnium, qui ex his α, β, γ, δ, ε, ζ, &c., vel ſumendis ſingulis, vel duobus pluribuſve in ſe multiplicandis, oriuntur, non excluſis æqualibus. Hoc ergo differt iſta numerorum Series ab illa, quæ (§.265.) prodiit, quod ibi Factores tantum diverſi ſumi debebant, hic autem idem Factor bis pluriesſve occurrere poſſit. Hic ſcilicet omnes numeri occurrunt, qui per multiplicationem ex his α, β, γ, δ, &c., provenire poſſunt.

272. Hanc ob rem Series ſemper ex terminorum numero infinito conſtat, ſive Factorum numerus fuerit infinitus, ſive finitus. Sic erit

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \&c.,$$

ubi omnes numeri adſunt, qui ex binario ſolo per multiplicationem oriuntur; ſeu omnes binarii Poteſtates. Deinde erit

$$\frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \&c.,$$

ubi

ubi alii numeri non occurrunt, niſi qui ex his duobus 2 & 3 per multiplicationem originem trahunt; ſeu qui alios Diviſores præter 2 & 3 non habent. CAP. XV.

273. Si igitur pro $\alpha, \beta, \gamma, \delta$, &c., unitas per ſingulos omnes numeros primos ſcribatur, ac ponatur

$$P = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1-\frac{1}{13})\ \&c.},$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi omnes numeri tam primi, quam qui ex primis per multiplicationem naſcuntur, occurrunt. Cum autem omnes numeri vel ſint ipſi primi, vel ex primis per multiplicationem oriundi, manifeſtum eſt, hic omnes omnino numeros integros in denominatoribus adeſſe debere.

274. Idem evenit, ſi numerorum primorum Poteſtates quæcunque accipiantur: ſi enim ponatur

$$P = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\&c.},$$

fiet

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

ubi omnes numeri naturales nullo excepto occurrunt. Quod ſi autem in Factoribus ubique ſignum + ſtatuatur, ut ſit

$$P = \frac{1}{(1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.},$$

erit

LIB. I.

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$$

ubi numeri primi habent ſignum $-$; qui ſunt producti ex duobus primis, ſive iiſdem ſive diverſis, ſignum habent $+$; & generatim, quorum numerorum numerus Factorum primorum eſt par, ſignum habent $+$, qui autem ex Factoribus primis numero imparibus conſtant, habent ſignum $-$. Sic terminus $\frac{1}{240^n}$, ob $240 = 2.2.2.2.3.5$, habebit ſignum $+$, cujus legis ratio percipitur ex §. 270, ſi ponatur $z = -1$.

275. Si hæc cum ſuperioribus conferantur, naſcentur binæ Series quarum productum unitati æquatur. Sit enim

$$P = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\ \&c.},$$

&

$$Q = (1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\ \&c.$$

erit

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

$$Q = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n}\ \&c.$$

(269), atque manifeſtum eſt fore $PQ = 1$.

275. Sin autem ponatur

$$P = \frac{1}{(1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.}$$

&

$Q =$

$$Q = (1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.,$$ CAP. XV.

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \&c.$$

$$Q = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

ſimilique modo habebitur $PQ = 1$. Cognita ergo alterius Seriei ſumma, ſimul alterius innoteſcet.

277. Viciſſim porro ex cognitis ſummis harum Serierum, aſſignari poterunt valores Factorum infinitorum. Sit nimirum

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.$$

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \frac{1}{7^{2n}} + \&c.,$$

eritque

$$M = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\ \&c.}$$

$$N = \frac{1}{(1-\frac{1}{2^{2n}})(1-\frac{1}{3^{2n}})(1-\frac{1}{5^{2n}})(1-\frac{1}{7^{2n}})(1-\frac{1}{11^{2n}})\ \&c.}$$

Hinc per diviſionem naſcitur

$$\frac{M}{N} = (1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.$$

 denique

denique vero erit

$$\frac{MM}{N} = \frac{2^n+1}{2^n-1} \cdot \frac{3^n+1}{3^n-1} \cdot \frac{5^n+1}{5^n-1} \cdot \frac{7^n+1}{7^n-1} \cdot \frac{11^n+1}{11^n-1} \cdot \&c.$$

Ex cognitis ergo M & N, præter valores horum productorum, ſummæ harum Serierum habebuntur

$$\frac{1}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \&c.$$

$$\frac{1}{N} = 1 - \frac{1}{2^{2n}} - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{6^{2n}} - \frac{1}{7^{2n}} + \frac{1}{10^{2n}} - \frac{1}{11^{2n}} - \&c.$$

$$\frac{M}{N} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

$$\frac{N}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$$

ex quarum combinatione multæ aliæ deduci poſſunt.

EXEMPLUM I.

Sit $n = 1$, &, quoniam ſupra demonſtravimus eſſe, $l\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c.$, erit, poſito $x = 1$, $l\frac{1}{1-1} = l\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$ At Logarithmus numeri infinite magni ∞ ipſe eſt infinite magnus, ex quo erit

$M =$

$$M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \&c. = \infty.$$ Cap. XV.

Hinc ob $\frac{1}{M} = \frac{1}{\infty} = 0$, fiet

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15} \;\&c..$$

Tum vero in productis habebitur

$$M = \infty = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\;\&c.,}$$

unde fit

$$\infty = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} \;\&c.,$$

&

$$0 = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{16}{17} \cdot \frac{18}{19} \cdot \;\&c..$$

Deinde per ſummationem Serierum ſupra traditam erit

$$N = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c. = \frac{\pi\pi}{6},$$ hinc obtinentur iſtæ ſummæ Serierum

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \&c.$$

$$\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \&c.$$

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} \;\&c.$$

Denique pro Factoribus orietur

LIB. I. $\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \&c.,$

ſeu

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{25}{24} \cdot \frac{49}{48} \cdot \frac{121}{120} \cdot \frac{169}{168} \cdot \&c.$$

&, ob $\frac{M}{N} = \infty$ ſeu $\frac{N}{M} = 0$, habebitur

$$\infty = \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \&c.,$$

ſeu

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \&c.,$$

atque

$$\infty = \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{6}{4} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{14}{12} \cdot \frac{18}{16} \cdot \frac{20}{18} \cdot \&c.,$$

ſeu

$$0 = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{9}{10} \cdot \&c.,$$

quarum fractionum (excepta prima) numeratores unitate deficiunt a denominatoribus , ſummæ autem ex numeratoribus & denominatoribus cujuſque fractionis conſtanter præbent numeros primos, 3, 5, 7, 11, 13, 17, 19, &c.

EXEMPLUM II.

Sit $n = 2$, eritque ex ſuperioribus

$$M = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c. = \frac{\pi\pi}{6}$$

$$N = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \&c. = \frac{\pi^4}{90}$$

Hinc primo iſtæ Series ſummantur

$\frac{6}{\pi\pi}$

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \&c.$$

CAP. XV.

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \&c.$$

$$\frac{15}{\pi^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{11^2} + \&c.$$

$$\frac{\pi\pi}{15} = 1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} \&c.,$$

Deinde valores ſequentium productorum innoteſcunt

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \&c.$$

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \&c.$$

$$\frac{15}{\pi\pi} = \frac{2^2+1}{2^2} \cdot \frac{3^2+1}{3^2} \cdot \frac{5^2+1}{5^2} \cdot \frac{7^2+1}{7^2} \cdot \frac{11^2+1}{11^2} \cdot \&c.,$$

ſeu

$$\frac{\pi\pi}{15} = \frac{4}{5} \cdot \frac{9}{10} \cdot \frac{25}{26} \cdot \frac{49}{50} \cdot \frac{121}{122} \cdot \frac{169}{170} \cdot \&c.,$$

&

$$\frac{5}{2} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \cdot \&c.,$$

ſive

$$\frac{5}{2} = \frac{5}{3} \cdot \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \&c.,$$

vel

$$\frac{3}{2} = \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \&c..$$

In his fractionibus numeratores unitate ſuperant denominatores, ſimul vero ſumti præbent quadrata numerorum primorum 3^2, 5^2, 7^2, 11^2, &c.

EXEM-

Exemplum III.

Quia ex ſuperioribus valores ipſius M tantum ſi n ſit numerus par, aſſignare licet, ponamus $n = 4$, eritque

$$M = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \&c. = \frac{\pi^4}{90}$$

$$N = 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \&c. = \frac{\pi^8}{9450}$$

Hinc primæ ſequentes Series ſummantur

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} \&c.$$

$$\frac{9450}{\pi^8} = 1 - \frac{1}{2^8} - \frac{1}{3^8} - \frac{1}{5^8} + \frac{1}{6^8} - \frac{1}{7^8} + \frac{1}{10^8} - \frac{1}{11^8} \&c.$$

$$\frac{105}{\pi^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{10^4} + \frac{1}{11^4} \&c.$$

$$\frac{\pi^4}{105} = 1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4} \&c..$$

Deinde etiam valores ſequentium productorum obtinentur

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \&c.$$

$$\frac{\pi^8}{9450} = \frac{2^8}{2^8-1} \cdot \frac{3^8}{3^8-1} \cdot \frac{5^8}{{}^8-1} \cdot \frac{7^8}{7^8-1} \cdot \frac{11^8}{11^8-1} \cdot \&c.$$

$$\frac{105}{\pi^4} = \frac{2^4+1}{2^4} \cdot \frac{3^4+1}{3^4} \cdot \frac{5^4+1}{5^4} \cdot \frac{7^4+1}{7^4} \cdot \frac{11^4+1}{11^4} \cdot \&c.$$

&

$$\frac{7}{6} = \frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7^4-1} \cdot \frac{11^4+1}{11^4-1} \cdot \&c.$$

ſeu

$$\frac{35}{34} = \frac{41}{40} \cdot \frac{313}{312} \cdot \frac{1201}{1200} \cdot \frac{7321}{7320} \cdot \&c.,$$

in

in his Factoribus numeratores unitate ſuperant denominatores, ſimul vero ſumti præbent bi-quadrata numerorum primorum imparium 3, 5, 7, 11, &c. CAP. XV.

278. Quoniam hic ſummam Seriei

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c.$$

ad Factores reduximus, ad Logarithmos commode progredi licebit. Nam, cum ſit

$$M = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\ \&c.},$$

erit

$$lM = -l(1-\frac{1}{2^n}) - l(1-\frac{3}{3^n}) - l(1-\frac{1}{5^n}) - l(1-\frac{1}{7^n}) - \&c..$$

Hinc, ſumendis Logarithmis hyperbolicis, erit

$$\begin{aligned} lM = &+1(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \&c.) \\ &+\frac{1}{2}(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c.) \\ &+\frac{1}{3}(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c.) \\ &+\frac{1}{4}(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c.) \\ &\&c. \end{aligned}$$

Quod ſi inſuper ponamus

LIB. I. $N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \&c.,$

ut sit

$$N = \frac{1}{(1-\frac{1}{2^{2n}})(1-\frac{1}{3^{2n}})(1-\frac{1}{5^{2n}})(1-\frac{1}{7^{2n}})(1-\frac{1}{11^{2n}})\ \&c.},$$

fiet, Logarithmis hyperbolicis sumendis,

$$\begin{aligned} lN = &+1\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c.\right) \\ &+\frac{1}{2}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c.\right) \\ &+\frac{1}{3}\left(\frac{1}{2^{6n}} + \frac{1}{3^{6n}} + \frac{1}{5^{6n}} + \frac{1}{7^{6n}} + \frac{1}{11^{6n}} + \&c.\right) \\ &+\frac{1}{4}\left(\frac{1}{2^{8n}} + \frac{1}{3^{8n}} + \frac{1}{5^{8n}} + \frac{1}{7^{8n}} + \frac{1}{11^{8n}} + \&c.\right) \\ &\&c. \end{aligned}$$

Ex his conjunctis fiet $lM - \frac{1}{2}\, lN =$

$$\begin{aligned} &+1\left(\frac{1}{2^{n}} + \frac{1}{3^{n}} + \frac{1}{5^{n}} + \frac{1}{7^{n}} + \frac{1}{11^{n}} + \&c.\right) \\ &+\frac{1}{3}\left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c.\right) \\ &+\frac{1}{5}\left(\frac{1}{2^{5n}} + \frac{1}{3^{5n}} + \frac{1}{5^{5n}} + \frac{1}{7^{5n}} + \frac{1}{11^{5n}} + \&c.\right) \\ &+\frac{1}{7}\left(\frac{1}{2^{7n}} + \frac{1}{3^{7n}} + \frac{1}{5^{7n}} + \frac{1}{7^{7n}} + \frac{1}{11^{7n}} + \&c.\right) \\ &\&c. \end{aligned}$$

279. Si

279. Si $n = 1$ erit $M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ CAP. XV.

$= l\,\infty$, & $N = \frac{\pi\pi}{6}$; hincque erit $l.\,l\,\infty - \frac{1}{2}\,l\,\frac{\varpi\varpi}{6} =$

$$+ 1\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \&c.\right)$$
$$+ \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} + \&c.\right)$$
$$+ \frac{1}{7}\left(\frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} + \&c.\right)$$
&c.

Verum hæ Series, præter primam, non ſolum ſummas habent finitas, ſed etiam cunctæ ſimul ſumtæ ſummam efficiunt finitam, eamque ſatis parvam: unde neceſſe eſt ut Seriei primæ $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c.$, ſumma ſit infinite magna, quantitate ſcilicet ſatis parva deficiet a Logarithmo hyperbolico Seriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$

280. Sit $n = 2$; erit $M = \frac{\pi\pi}{6}$ & $N = \frac{\varpi^4}{90}$: unde fit

$$2l\pi - l6 = 1\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.\right)$$
$$+ \frac{1}{2}\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c.\right)$$
$$+ \frac{1}{2}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c.\right)$$
&c.

$$4l\pi - l90 = +1\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c.\right)$$
$$+\frac{1}{2}\left(\frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{11^8} + \&c.\right)$$
$$+\frac{1}{3}\left(\frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \frac{1}{11^{12}} + \&c.\right)$$
&c.

$$\frac{1}{2}l\frac{5}{2} = 1\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.\right)$$
$$+\frac{1}{3}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c.\right)$$
$$+\frac{1}{5}\left(\frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \frac{1}{11^{10}} + \&c.\right)$$
&c.

281. Quanquam lex, qua numeri primi progrediuntur, non constat, tamen harum Serierum altiorum Potestatum summæ non difficulter proxime assignari poterunt. Sit enim hæc Series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.,$$
&
$$S = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \&c.$$
erit
$$S = M - 1 - \frac{1}{4^n} - \frac{1}{6^n} - \frac{1}{8^n} - \frac{1}{9^n} - \frac{1}{10^n} - \&c.,$$
& ob
$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \frac{1}{12^n} + \&c.,$$
erit
$$S = M - \frac{M}{2^n} - 1 + \frac{1}{2^n} - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \&c.;$$
seu

$S =$

$$S = (M - 1)\left(1 - \frac{1}{2^n}\right) - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \frac{1}{25^n} - \frac{1}{27^n} - \&c.,$$ CAP. XV.

& ob

$$M\left(1 - \frac{1}{2^n}\right)\frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \&c.,$$

erit

$$S = (M - 1)\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right) + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{45^n} - \&c..$$

Hinc, ob datam summam M, valor ipsius S commode invenitur, si quidem n fuerit numerus mediocriter magnus.

282. Inventis autem summis altiorum Potestatum, etiam summæ Potestatum minorum ex formulis inventis exhiberi possunt. Atque hac methodo sequentes prodierunt summæ Seriei

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

si sit	erit summa Seriei
$n = 2$;	0, 452247420041222
$n = 4$;	0, 076993139764252
$n = 6$;	0, 017070086850639
$n = 8$;	0, 004061405366515
$n = 10$;	0, 000993603573633
$n = 12$;	0, 000246026470033
$n = 14$;	0, 000061244396725
$n = 16$;	0, 000015282026219
$n = 18$;	0, 000003817278702

$n = 20$;	0, 000000953961123
$n = 22$;	0, 000000238450446
$n = 24$;	0, 000000059608184
$n = 26$;	0, 000000014901555
$n = 28$;	0, 000000003725333
$n = 30$;	0, 000000000931323
$n = 32$;	0, 000000000232830
$n = 34$;	0, 000000000058207
$n = 36$;	0, 000000000014551

reliquæ ſummæ parium Poteſtatum in ratione quadrupla decreſcunt.

283. Hæc autem Seriei $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} +$ &c., in productum infinitum converſio etiam directe inſtitui poteſt hoc modo: ſit

$$A = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{\&c., ſubtrahe}$$

$$\frac{1}{2^n} A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \text{\&c.,}$$

erit

$$\left(1 - \frac{1}{2^n}\right) A = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{\&c.}$$

$= B$: ſic ſublati ſunt omnes termini per 2 diviſibiles,

ſubtr. $\frac{1}{3^n} B = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} +$ &c.,

erit

$$\left(1 - \frac{1}{3^n}\right) B = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \text{\&c.} = C:$$

ſic inſuper ſublati ſunt omnes termini per 3 diviſibiles,

ſubtr.

subtr. $\frac{1}{5^n} C = \quad \frac{1}{5^n} + \frac{1}{25^n} + \frac{1}{35^n} + \frac{1}{55^n} + \&c.,$

erit

$$(1 - \frac{1}{5^n}) C = 1 + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

sic sublati etiam sunt omnes termini per 5 divisibiles. Pari modo tolluntur termini divisibiles per 7, 11, reliquosque numeros primos; manifestum autem est sublatis omnibus terminis, qui per numeros primos divisibiles sint, solam unitatem relinqui. Quare pro B, C, D, E, &c., valoribus restitutis tandem orietur

$$A(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c. = 1,$$

unde Seriei propositæ summa erit =

$$A = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c.,}$$

seu

$$A = \frac{2^n}{2^n - 1} \cdot \frac{3^n}{3^n - 1} \cdot \frac{5^n}{5^n - 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n - 1} \cdot \&c.$$

284. Hæc methodus jam commode adhiberi poterit ad alias Series, quarum summas supra invenimus, in producta infinita convertendas. Invenimus autem supra (175.) summas harum Serierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \&c.,$$

si n fuerit numerus impar, summa enim est $= N\pi^n$ & valores ipsius N loco citato dedimus. Notandum autem est

cum

LIB. I. cum hic tantum numeri impares occurrunt, eos qui sint formæ $4m+1$ habere signum $+$, reliquos formæ $4m-1$ signum $-$. Sit igitur

$$A = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{15^n} + \&c.$$

$$\frac{1}{3^n}A = \frac{1}{3^n} - \frac{1}{9^n} + \frac{1}{15^n} - \frac{1}{21^n} + \frac{1}{27^n} - \&c.,$$ addatur, erit

$$\left(1+\frac{1}{3^n}\right)A = 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n}\ \&c., = B$$

$$\frac{1}{5^n}B = \frac{1}{5^n} + \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{55^n}\ \&c.,$$ subtrahatur, erit

$$\left(1-\frac{1}{5^n}\right)B = 1 - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = C,$$

ubi jam numeri per 3 & 5 divisibiles desunt,

$$\frac{1}{7^n}C = \frac{1}{7^n} - \frac{1}{49^n} - \frac{1}{77^n} + \&c.,$$ addatur, erit

$$\left(1+\frac{1}{7^n}\right)C = 1 - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = D,$$

sic etiam numeri per 7 divisibiles sunt sublati

$$\frac{1}{11^n}D = \frac{1}{11^n} - \frac{1}{121^n} + \&c.,$$ addatur, erit

$$\left(1+\frac{1}{11^n}\right)D = 1 + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = E$$

sic numeri per 11 divisibiles quoque sunt sublati. Auferendis autem

autem hoc modo reliquis numeris omnibus per reliquos numeros primos divisibilibus, tandem prodibit

$$A\left(1+\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\left(1-\frac{1}{13^n}\right) \&c. = 1,$$

seu

$$A = \frac{3^n}{3^n+1}\cdot\frac{5^n}{5^n-1}\cdot\frac{7^n}{7^n+1}\cdot\frac{11^n}{11^n+1}\cdot\frac{13^n}{13^n-1}\times\frac{17^n}{17^n-1}.\&c.,$$

ubi in numeratoribus occurrunt Potestates omnium numerorum primorum, quæ in denominatoribus insunt unitate sive auctæ sive minutæ, prout numeri primi fuerint formæ $4m-1$, vel $4m+1$.

285. Posito ergo $n=1$, ob $A=\frac{\pi}{4}$, erit

$$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}.\ \&c.,$$

supra autem invenimus esse

$$\frac{\pi\pi}{6} = \frac{4}{3}\cdot\frac{3^2}{2.4}\cdot\frac{5^2}{4.6}\cdot\frac{7^2}{6.8}\cdot\frac{11^2}{10.12}\cdot\frac{13^2}{12.14}\cdot\frac{17^2}{16.18}\cdot\frac{19^2}{18.20}.\ \&c.$$

Dividatur secunda per primam & orietur

$$\frac{2\pi}{3} = \frac{4}{3}\cdot\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}.\ \&c.,$$

seu

$$\frac{\pi}{2} = \frac{2}{3}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}.\ \&c.,$$

ubi numeri primi constituunt numeratores, denominatores vero sunt numeri impariter pares, unitate differentes a nume-

LIB. I. ratoribus. Quod si hæc denuo per primam $\frac{\varpi}{4}$ dividatur, erit

$$2 = \frac{4}{2} \cdot \frac{4}{6} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{16}{18} \cdot \frac{20}{18} \cdot \frac{24}{22} \cdot \&c.,$$

seu

$$2 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdot \&c.,$$

quæ fractiones oriuntur ex numeris primis imparibus 3, 5, 7, 11, 13, 17, &c., quemque in duas partes unitate differentes dispescendo, & partes pares pro numeratoribus, impares pro denominatoribus sumendo.

286. Si hæ expressiones cum *Wallisiana* comparentur

$$\frac{\varpi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11}\ \&c.,$$

seu

$$\frac{4}{\pi} = \frac{3.3}{2.4} \cdot \frac{5.5}{4.6} \cdot \frac{7.7}{6.8} \cdot \frac{9.9}{8.10} \cdot \frac{11.11}{10.12} \cdot \&c.,$$

cum sit

$$\frac{\pi\pi}{8} = \frac{3.3}{2.4} \cdot \frac{5.5}{4.6} \cdot \frac{7.7}{6.8} \cdot \frac{11.11}{10.12} \cdot \frac{13.13}{12.14} \cdot \&c.,$$

illa per hanc divisa dabit

$$\frac{32}{\varpi^3} = \frac{9.9}{8.10} \cdot \frac{15.15}{14.16} \cdot \frac{21.21}{20.22} \cdot \frac{25.25}{24.26} \cdot \&c.,$$

ubi in numeratoribus occurrunt omnes numeri impares non primi.

287. Sit jam $n = 3$ erit $A = \frac{\pi^3}{32}$, unde fit

$$\frac{\pi^3}{32} = \frac{3^3}{3^3+1} \cdot \frac{5^3}{5^3-1} \cdot \frac{7^3}{7^3+1} \cdot \frac{11^3}{11^3+1} \cdot \frac{13^3}{13^3-1} \cdot \frac{17^3}{17^3-1} \cdot$$

&c.. At ex Serie

$$\frac{\pi^6}{945} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \&c.,$$

fit

$\frac{\varpi^6}{945}$

$$\frac{\pi^6}{945} = \frac{2^6}{2^6-1} \cdot \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot$$

&c., seu

$$\frac{\pi^6}{960} = \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot \&c.,$$

quæ per primam divisa dabit

$$\frac{\pi^3}{30} = \frac{3^3}{3^3-1} \cdot \frac{5^3}{5^3+1} \cdot \frac{7^3}{7^3-1} \cdot \frac{11^3}{11^3-1} \cdot \frac{13^3}{13^3+1} \cdot \frac{17^3}{17^3+1} \cdot$$

&c.,

hæc vero denuo per primam divisa dabit

$$\frac{16}{15} = \frac{3^3+1}{3^3-1} \cdot \frac{5^3-1}{5^3+1} \cdot \frac{7^3+1}{7^3-1} \cdot \frac{11^3+1}{11^3-1} \cdot \frac{13^3-1}{13^3+1} \cdot \frac{17^3-1}{17^3+1} \cdot$$

&c., seu

$$\frac{16}{15} = \frac{14}{13} \cdot \frac{62}{63} \cdot \frac{172}{171} \cdot \frac{666}{665} \cdot \frac{1098}{1099} \cdot \&c.,$$

quæ fractiones formantur ex cubis numerorum primorum imparium, quemque in duas partes unitate differentes dispescendo, ac partes pares pro numeratoribus, impares pro denominatoribus sumendo.

288. Ex his expressionibus denuo novæ Series formari possunt, in quibus omnes numeri naturales denominatores constituunt. Cum enim sit

$$\frac{\pi}{4} = \frac{3}{3+1} \cdot \frac{5}{5-1} \cdot \frac{7}{7+1} \cdot \frac{11}{11+1} \cdot \frac{13}{13-1} \cdot \&c.,$$

erit

$$\frac{\pi}{6} = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\ \&c.},$$

unde per evolutionem hæc Series nascetur

$$\frac{\pi}{6} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\ \&c,$$

 ubi

LIB. I. ubi ratio ſignorum ita eſt comparata, ut binarius habeat —; numeri primi formæ $4m - 1$ ſignum —; & numeri primi formæ $4m + 1$ ſignum +; numeri autem compoſiti ea habent ſigna, quæ ipſis ratione multiplicationis ex primis conveniunt. Sic patebit ſignum fractionis $\frac{1}{60}$, ob $60 = \overset{-}{2} . \overset{-}{2} . \overset{-}{3} . \overset{+}{5}$, quod erit —. Simili modo porro erit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\ \&c.},$$

unde orietur hæc Series

$$\frac{\pi}{2} = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\ \&c.,$$

ubi binarius habet ſignum +; numeri primi formæ $4m - 1$ ſignum —; numeri primi formæ $4m + 1$ ſignum +; & numerus quiſque compoſitus id habet ſignum, quod ipſi ratione compoſitionis ex primis convenit, ſecundum regulas multiplicationis.

289. Cum deinde ſit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\ \&c.},$$

erit per evolutionem

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15}\ \&c.$$

ubi tantum numeri impares occurrunt, ſigna autem ita ſunt comparata, ut numeri primi formæ $4m - 1$ ſignum habeant +; numeri primi formæ $4m + 1$ ſignum —; unde ſimul numerorum compoſitorum ſigna definiuntur. Binæ porro Series hinc formari poſſunt, ubi omnes numeri occurrunt, erit ſcilicet

$\pi =$

CAP. XV.

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\ \&c.},$$

unde per evolutionem oritur

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\ \&c.,$$

ubi binarius ſignum habet $+$; numeri primi formæ $4m-1$ ſignum $+$; numeri vero primi formæ $4m+1$ ſignum $-$. Tum vero etiam erit

$$\frac{\pi}{3} = \frac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\ \&c.},$$

unde per evolutionem oritur

$$\frac{\pi}{3} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\ \&c.,$$

ubi binarius habet ſignum $-$, numeri primi formæ $4m-1$ ſignum $+$, & numeri primi formæ $4m+1$ ſignum $-$.

290. Poſſunt hinc etiam innumerabiles aliæ ſignorum conditiones exhiberi, ita ut Seriei

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \&c.,$$

ſumma aſſignari queat. Cum ſcilicet ſit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\ \&c.}.$$

 Multi-

LIB. I.

Multiplicetur hæc expressio per $\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$, erit

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\ \&c.},$$

&

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11}\ \&c.,$$

ubi binarius signum habet $+$; ternarius $+$; reliqui numeri primi omnes formæ $4m - 1$ signum $-$; at numeri primi formæ $4m + 1$ signum $+$; & unde pro numeris compositis ratio signorum intelligitur. Simili modo, cum sit

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\ \&c.},$$

multiplicetur per $\frac{1+\frac{1}{5}}{1-\frac{1}{5}} = \frac{3}{2}$, erit

$$\frac{3\pi}{2} = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})(1+\frac{1}{17})\ \&c.},$$

unde per evolutionem oritur

$$\frac{3\pi}{2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi binarius habet signum $+$; numeri primi formæ $4m - 1$ signum $+$; & numeri primi formæ $4m + 1$, præter quinarium, signum $-$.

291. Pos-

291. Possunt etiam innumerabiles hujusmodi Series exhiberi, quarum summa sit $= 0$. Cum enim sit CAP. XV.

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \&c.,$$

erit

$$0 = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\ \&c.},$$

unde, ut supra vidimus, oritur

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\ \&c.,$$

ubi omnes numeri primi signum habent $-$; compositorumque numerorum signa regulam multiplicationis sequuntur. Multiplicemus autem illam expressionem per $\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$, erit pariter

$$0 = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\ \&c.}$$

unde per evolutionem nascitur

$$0 = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\ \&c.,$$

ubi binarius habet signum $+$; reliqui numeri primi omnes signum $-$. Simili modo quoque erit

$0 =$

Lib. I.

$$0 = \frac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\ \&c.},$$

unde oritur iſta Series

$$0 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\ \&c.,$$

ubi omnes numeri primi, præter 3 & 5, habent ſignum —. In genere autem notandum eſt, quoties omnes numeri primi, exceptis tantum aliquibus, habeant ſignum —, ſummam Seriei fore = 0. Contra autem quoties omnes numeri primi, exceptis tantum aliquibus, habeant ſignum +, tum ſummam Seriei fore infinite magnam.

292. Supra etiam (176.) ſummam dedimus Seriei

$$A = 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \frac{1}{11^n} + \frac{1}{13^n}\ \&c.,$$

ſi fuerit n numerus impar: Erit ergo

$$\frac{1}{2^n}A = \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{8^n} - \frac{1}{10^n} + \frac{1}{14^n} - \&c.,$$

quæ addita dat

$$B = (1 + \frac{1}{2^n})A = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \frac{1}{25^n} - \&c.$$

$$\frac{1}{5^n}B = \frac{1}{5^n} - \frac{1}{25^n} + \frac{1}{35^n} - \frac{1}{55^n}\ \&c.,$$ addatur, erit

$C =$

$$C = \left(1 + \frac{1}{5^n}\right)B = 1 + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} \text{ \&c.}$$

CAP. XV.

$\frac{1}{7^n}C = \frac{1}{7^n} + \frac{1}{49^n} - \frac{1}{77^n} + \text{ \&c.}$, ſubtrahatur, erit

$$D = \left(1 - \frac{1}{7^n}\right)C = 1 - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \text{ \&c.}.$$

Ex his tandem fiet

$$A\left(1 + \frac{1}{2^n}\right)\left(1 + \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 + \frac{1}{11^n}\right)\left(1 - \frac{1}{13^n}\right) \text{\&c.} = 1,$$

ubi numeri primi unitate excedentes multipla ſenarii habent ſignum $-$, deficientes autem ſignum $+$. Eritque

$$A = \frac{2^n}{2^n + 1} \cdot \frac{5^n}{5^n + 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n + 1} \cdot \frac{13^n}{13^n - 1} \cdot \text{ \&c.},$$

293. Conſideremus caſum $n = 1$, quo $A = \frac{\pi}{3\sqrt{3}}$, eritque

$$\frac{\pi}{3\sqrt{3}} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \text{ \&c.},$$

ubi in numeratoribus poſt 3 occurrunt omnes numeri primi, denominatores vero a numeratoribus unitate diſcrepant, ſuntque omnes per 6 diviſibiles. Cum jam ſit

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{11 \cdot 11}{10 \cdot 12} \cdot \frac{13 \cdot 13}{12 \cdot 14} \text{ \&c.},$$

erit, hac expreſſione per illam diviſa,

$$\frac{\pi\sqrt{3}}{2} = \frac{9}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \text{ \&c.},$$

LIB. I. ubi denominatores non ſunt per 6 diviſibiles. Vel erit

$$\frac{\pi}{2\sqrt{3}} = \frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{24}\cdot \&c.$$

$$\frac{2\pi}{3\sqrt{3}} = \frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{22}\cdot \&c.,$$

quarum hæc per illam diviſa dat

$$\frac{4}{3} = \frac{6}{4}\cdot\frac{6}{8}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{18}{16}\cdot\frac{18}{20}\cdot\frac{24}{22}\cdot \&c.,$$

ſeu

$$\frac{4}{3} = \frac{3}{2}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{9}{8}\cdot\frac{9}{10}\cdot\frac{12}{11}\cdot \&c.,$$

ubi ſingulæ fractiones ex numeris primis 5, 7, 11, &c., formantur, ſingulos numeros primos in duas partes unitate differentes diſpeſcendo, & partes per 3 diviſibiles conſtanter pro numeratoribus ſumendo.

294. Quoniam vero ſupra vidimus eſſe

$$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot \&c.,$$

ſeu

$$\frac{\pi}{3} = \frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot \&c.,$$

ſi ſuperiores $\frac{\pi}{2\sqrt{3}}$ & $\frac{2\pi}{3\sqrt{3}}$ per hanc dividantur, orietur

$$\frac{\sqrt{3}}{2} = \frac{2}{3}\cdot\frac{4}{3}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{14}{15}\cdot\frac{16}{15}\cdot \&c.$$

$$\frac{2}{\sqrt{3}} = \frac{6}{5}\cdot\frac{6}{7}\cdot\frac{12}{11}\cdot\frac{18}{19}\cdot\frac{24}{23}\cdot\frac{30}{29}\cdot \&c.,$$

In priori expreſſione fractiones formantur ex numeris primis formæ $12m + 6 \pm 1$, in poſteriore ex numeris primis formæ $12m \pm 1$, ſingulos in duas partes unitate diſcrepantes diſpeſcendo, & partes pares pro numeratoribus, impares vero pro denominatoribus ſumendo.

295. Contemplemur adhuc Seriem ſupra inventam (179), quæ ita progrediebatur

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{\&c.} = A, \text{ erit}$$

$$\frac{1}{3}A = \frac{1}{3} + \frac{1}{9} - \frac{1}{15} - \frac{1}{21} + \frac{1}{27} + \frac{1}{33} - \text{\&c.: ſubtrahatur}$$

$$(1 - \frac{1}{3})A = 1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} + \frac{1}{19} - \text{\&c.} = B$$

$$\frac{1}{5}B = \frac{1}{5} - \frac{1}{25} - \frac{1}{35} + \frac{1}{55} - \text{\&c.:addatur, erit}$$

$$(1 + \frac{1}{5})B = 1 - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} \text{ \&c.} = C:$$

ſicque progrediendo tandem pervenietur ad

$$\frac{\pi}{2\sqrt{2}}(1 - \frac{1}{3})(1 + \frac{1}{5})(1 + \frac{1}{7})(1 - \frac{1}{11})(1 + \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19}) \text{ \&c.} = 1.$$

ubi ſigna ita ſe habent, ut numerorum primorum formæ $8m + 1$, vel $8m + 3$, ſigna ſint $-$; numerorum primorum vero formæ $8m + 5$, vel $8m + 7$, ſigna ſint $+$. Hinc itaque erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \text{\&c.},$$

ubi omnes denominatores vel diviſibiles ſunt per 8, vel tantum ſunt numeri impariter pares. Cum igitur ſit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \text{\&c.}$$

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \text{\&c.},$$

&

$$\frac{\pi\pi}{8} = \frac{3.3}{2.4} \cdot \frac{5.5}{4.6} \cdot \frac{7.7}{6.8} \cdot \frac{11.11}{10.12} \cdot \frac{13.13}{12.14} \cdot \&c.,$$

erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \&c.,$$

ubi nulli denominatores per 8 divisibiles occurrunt, pariter pares vero adsunt, quoties unitate differunt a numeratoribus. Prima vero per ultimam divisa dat

$$1 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{10}{9} \cdot \frac{11}{12} \cdot \&c.,$$

quæ fractiones formantur ex numeris primis, singulos in duas partes unitate discrepantes dispescendo, & partes pares, (nisi sint pariter pares) pro numeratoribus sumendo.

296. Simili modo reliquæ Series, quas supra pro expressione arcuum circularium invenimus (179. *& seqq.*) in Factores transformari possunt, qui ex numeris primis constituantur. Sicque multæ aliæ insignes proprietates tam hujusmodi Factorum, quam Serierum infinitarum erui poterunt. Quoniam vero præcipuas hic jam commemoravi, pluribus evolvendis hic non immorabor. Sed ad aliud huic affine argumentum procedam. Quemadmodum scilicet in hoc Capite numeri, quatenus per multiplicationem oriuntur, sunt considerati, ita in sequenti generatio numerorum per additionem perpendetur.

CAPUT

CAPUT XVI.

De Partitione numerorum.

297. PRoposita sit ista expressio

$$(1+x^{\alpha}z)(1+x^{\beta}z)(1+x^{\gamma}z)(1+x^{\delta}z)(1+x^{\epsilon}z)\,\&c.,$$

quæ cujusmodi induat formam, si per multiplicationem evolvatur, inquiramus. Ponamus prodire

$$1+Pz+Qz^{2}+Rz^{3}+Sz^{4}+\&c.,$$

atque manifestum est P fore summam Potestatum

$x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\epsilon}+\&c.$. Deinde Q est summa Factorum ex binis Potestatibus diversis, seu Q erit aggregatum plurium Potestatum ipsius x, quarum Exponentes sunt summæ duorum terminorum diversorum hujus Seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \eta,\ \&c.$$

Simili modo R erit aggregatum Potestatum ipsius x, quarum Exponentes sunt summæ trium terminorum diversorum. Atque S erit aggregatum Potestatum ipsius x, quarum Exponentes sunt summæ quatuor terminorum diversorum ejusdem Seriei, $\alpha, \beta, \gamma, \delta, \epsilon$, &c., & ita porro.

298. Singulæ hæ Potestates ipsius x, quæ in valoribus literarum P, Q, R, S, &c., insunt, unitatem pro coëfficiente habebunt, si quidem earum Exponentes unico modo ex

$\alpha, \beta, \gamma, \delta$, &c., formari queant: ſin autem ejuſdem Poteſtatis Exponens pluribus modis poſſit eſſe ſumi duorum, trium, pluriumve terminorum Seriei $\alpha, \beta, \gamma, \delta, \varepsilon$, &c., tum etiam Poteſtas illa coëfficientem habebit, qui unitatem toties in ſe complectatur. Sic, ſi in valore ipſius Q reperiatur Nx^n, indicio hoc erit numerum n eſſe N diverſis modis ſummam duorum terminorum diverſorum Seriei α, β, γ, &c.. Atque ſi in evolutione Factorum propoſitorum occurrat terminus $Nx^n z^m$, ejus coëfficiens N indicabit quot variis modis numerus n poſſit eſſe ſumma m terminorum diverſorum Seriei $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, &c.

299. Quod ſi ergo productum propoſitum

$$(1 + x^\alpha z)(1 + x^\beta z)(1 + x^\gamma z)(1 + x^\delta z) \text{ \&c.},$$

per multiplicationem veram evolvatur, ex expreſſione reſultante ſtatim apparebit, quot variis modis datus numerus poſſit eſſe ſumma tot terminorum diverſorum Seriei $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, &c., quot quis voluerit. Scilicet, ſi quæratur quot variis modis numerus n poſſit eſſe ſumma m terminorum illius Seriei diverſorum, in expreſſione evoluta quæri debet terminus $x^n z^m$, ejuſque coëfficiens indicabit numerum quæſitum.

300. Quo hæc fiant planiora, ſit propoſitum hoc productum ex Factoribus conſtans infinitis

$$(1 + xz)(1 + x^2 z)(1 + x^3 z)(1 + x^4 z)(1 + x^5 z) \text{ \&c.},$$

quod per multiplicationem actualem evolutum dat

$$
\begin{array}{l}
1+z\,(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.)\\
+z^2\,(x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+4x^{10}+5x^{11}+\&c.)\\
+z^3\,(x^6+x^7+2x^8+3x^9+4x^{10}+5x^{11}+7x^{12}+8x^{13}+10x^{14}+\&c.)\\
+z^4\,(x^{10}+x^{11}+2x^{12}+3x^{13}+5x^{14}+6x^{15}+9x^{16}+11x^{17}+15x^{18}+\&c.)\\
+z^5\,(x^{15}+x^{16}+2x^{17}+3x^{18}+5x^{19}+7x^{20}+10x^{21}+13x^{22}+18x^{23}+\&c.)\\
+z^6\,(x^{21}+x^{22}+2x^{23}+3x^{24}+5x^{25}+7x^{26}+11x^{27}+14x^{28}+20x^{29}+\&c.)\\
+z^7\,(x^{28}+x^{29}+2x^{30}+3x^{31}+5x^{32}+7x^{33}+11x^{34}+15x^{35}+21x^{36}+\&c.)\\
+z^8\,(x^{36}+x^{37}+2x^{38}+3x^{39}+5x^{40}+7x^{41}+11x^{42}+15x^{43}+22x^{44}+\&c.)\\
\qquad\qquad\qquad\&c.
\end{array}
$$

CAP. XVI.

Ex his ergo Seriebus ſtatim definire licet quot variis modis propoſitus numerus ex dato terminorum diverſorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, 8, &c., numero oriri queat. Sic, ſi quæratur quot variis modis numerus 35 poſſit eſſe ſumma ſeptem terminorum diverſorum Seriei 1, 2, 3, 4, 5, 6, 7, &c., quæratur in Serie z^7 multiplicante Poteſtas x^{35}, ejuſque coëfficiens 15 indicabit numerum propoſitum 35 quindecim variis modis eſſe ſummam ſeptem terminorum Seriei 1, 2, 3, 4, 5, 6, 7, 8, &c.

301. Quod ſi autem ponatur $z = 1$, & ſimiles Poteſtates ipſius x in unam ſummam conjiciantur, ſeu, quod eodem redit, ſi evolvatur hæc expreſſio infinita

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\&c.,$$

quo facto orietur hæc Series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+\&c.,$$

ubi quivis coëfficiens indicat, quot variis modis Exponens Poteſtatis ipſius x conjunctæ ex terminis diverſis Seriei 1, 2, 3, 4, 5, 6, 7, &c., per additionem emergere poſſit. Sic apparet numerum 8 ſex modis per additionem diverſorum numerorum produci, qui ſunt

$8 = 8$	$8 = 5+3$
$8 = 7+1$	$8 = 5+2+1$
$8 = 6+2$	$8 = 4+3+1$

ubi

LIB. I. ubi notandum est numerum propositum ipsum simul computari debere, quia numerus terminorum non definitur, ideoque unitas inde non excluditur.

302. Hinc igitur intelligitur, quomodo quisque numerus per additionem diversorum numerorum producatur. Conditio autem diversitatis omittetur, si Factores illos in denominatorem transponamus. Sit igitur proposita hæc expressio

$$\frac{1}{(1-x^{\alpha}z)(1-x^{\beta}z)(1-x^{\gamma}z)(1-x^{\delta}z)(1-x^{\epsilon}z)\,\&c.},$$

quæ per divisionem evoluta det

$$1+Pz+Qz^2+Rz^3+Sz^4+\&c..$$

Atque manifestum est fore P aggregatum Potestatum ipsius x, quarum Exponentes contineantur in hac Serie

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \eta,\ \&c.,$$

Deinde Q erit aggregatum Potestatum ipsius x, quarum Exponentes sint summæ duorum terminorum hujus Seriei, sive eorundem sive diversorum. Tum erit R summa Potestatum ipsius x, quarum Exponentes ex additione trium terminorum illius Seriei oriantur; & S summa Potestatum, quarum Exponentes ex additione quatuor terminorum in illa Serie contentorum formantur, & ita porro.

303. Si igitur tota expressio per singulos terminos explicetur, & termini similes conjunctim exprimantur, intelligetur quot variis modis propositus numerus n per additionem m terminorum, sive diversorum sive non diversorum, Seriei α, β, γ, δ, ϵ, ζ, &c., produci queat. Quæratur scilicet in expressione evoluta terminus $x^n z^m$, ejusque coëfficiens, qui sit N, ita ut totus terminus sit $= Nx^n z^m$, atque coëfficiens N indicabit quot variis modis numerus n per additionem m terminorum

minorum in Serie $\alpha, \beta, \gamma, \delta, \varepsilon$, &c., contentorum produci queat. Hoc igitur pacto quæſtio priori, quam ante ſumus contemplati, ſimilis reſolvetur. CAP. XVI.

304. Accommodemus hæc ad caſum inprimis notatu dignum, ſitque propoſita hæc expreſſio

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\,\&c.},$$

quæ per diviſionem evoluta dabit

$$\begin{array}{l}
1+z(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.)\\
+z^2(x^2+x^3+2x^4+2x^5+3x^6+3x^7+4x^8+4x^9+5x^{10}+\&c.)\\
+z^3(x^3+x^4+2x^5+3x^6+4x^7+5x^8+7x^9+8x^{10}+10x^{11}+\&c.)\\
+z^4(x^4+x^5+2x^6+3x^7+5x^8+6x^9+9x^{10}+11x^{11}+15x^{12}+\&c.)\\
+z^5(x^5+x^6+2x^7+3x^8+5x^9+7x^{10}+10x^{11}+13x^{12}+18x^{13}+\&c.)\\
+z^6(x^6+x^7+2x^8+3x^9+5x^{10}+7x^{11}+11x^{12}+14x^{13}+20x^{14}+\&c.)\\
+z^7(x^7+x^8+2x^9+3x^{10}+5x^{11}+7x^{12}+11x^{13}+15x^{14}+21x^{15}+\&c.)\\
+z^8(x^8+x^9+2x^{10}+3x^{11}+5x^{12}+7x^{13}+11x^{14}+15x^{15}+22x^{16}+\&c.)\\
\&c.,
\end{array}$$

Ex his ergo Seriebus ſtatim definire licet quot variis modis propoſitus numerus per additionem ex dato terminorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, &c., numero produci queat. Sic, ſi quæratur quot variis modis numerus 13 oriri poſſit per additionem quinque numerorum integrorum, ſpectari debebit terminus $x^{13}z^5$, cujus coëfficiens 18 indicat numerum propoſitum 13 ex quinque numerorum additione octodecim modis oriri poſſe.

305. Si ponatur $z = 1$, atque ſimiles Poteſtates ipſius x conjunctim exprimantur, hæc expreſſio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\,\&c.},$$

evolvetur in hanc Seriem

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 \text{ \&c.},$$

in qua quilibet coëfficiens indicat, quot variis modis Exponens Poteſtatis adjunctæ per additionem produci queat ex numeris integris, ſive æqualibus ſive inæqualibus. Scilicet ex termino $11x^6$ cognoſcitur numerum 6 undecim modis par additionem numerorum integrorum produci poſſe, qui ſunt

$6 = 6$	$6 = 3 + 1 + 1 + 1$
$6 = 5 + 1$	$6 = 2 + 2 + 2$
$6 = 4 + 2$	$6 = 2 + 2 + 1 + 1$
$6 = 4 + 1 + 1$	$6 = 2 + 1 + 1 + 1 + 1$
$6 = 3 + 3$	$6 = 1 + 1 + 1 + 1 + 1 + 1$
$6 = 3 + 2 + 1$	

ubi quoque notari debet, ipſum numerum propoſitum, cum in Serie numerorum 1, 2, 3, 4, 5, 6, &c., propoſita contineatur, unum modum præbere.

306. His in genere expoſitis, diligentius inquiramus in modum hanc compoſitionum multitudinem inveniendi. Ac primo quidem conſideremus eam ex numeris integris compoſitionem, in qua numeri tantum diverſi admittuntur, quam prius commemoravimus. Sit igitur in hunc finem propoſita hæc expreſſio

$$Z = (1 + xz)(1 + x^2z)(1 + x^3z)(1 + x^4z)(1 + x^5z) \text{ \&c.},$$

quæ evoluta & ſecundum Poteſtates ipſius z digeſta præbeat

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \text{\&c.},$$

ubi methodus deſideratur has ipſius x Functiones P, Q, R, S, T, &c., expedite inveniendi, hoc enim pacto quæſtioni propoſitæ convenientiſſime ſatisfiet.

307. Patet autem, ſi loco z ponatur xz, prodire

(1 +

$(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)$ &c. $= \frac{Z}{1+xz}$ CAP. XVI.

ergo, posito xz loco z, valor producti, qui erat Z, abibit in $\frac{Z}{1+xz}$; sicque, cum sit

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.,$$

erit

$$\frac{Z}{1+xz} = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c.,$$

multiplicetur ergo actu per $1+xz$, atque prodibit

$$\begin{aligned} Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c. \\ + xz + Px^2z^2 + Qx^3z^3 + Rx^4z^4 + \&c., \end{aligned}$$

qui valor ipsius Z cum superiori comparatus dabit

$$P = \frac{x}{1-x};\ Q = \frac{Px^2}{1-x^2};\ R = \frac{Qx^3}{1-x^3};\ S = \frac{Rx^4}{1-x^4}\ \&c.,$$

Sequentes ergo pro P, Q, R, S, &c., obtinentur valores

$$P = \frac{x}{1-x}$$

$$Q = \frac{x^3}{(1-x)(1-x^2)}$$

$$R = \frac{x^6}{(1-x)(1-x^2)(1-x^3)}$$

$$S = \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

$$T = \frac{x^{15}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$$

&c.

308. Sic igitur seorsim unamquamque Seriem Potestatum ipsius x exhibere possumus, ex qua definire licet, quot variis modis propositus numerus ex dato partium integrarum numero per additionem formari possit. Manifestum autem porro est has singulas Series esse recurrentes, quia ex evolutione Functionis fractæ ipsius x nascuntur. Prima scilicet expressio

LIB. I. $P = \frac{x}{1-x}$, dat Seriem geometricam

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c.,$$

ex qua quidem manifestum est quemvis numerum semel in Serie numerorum integrorum contineri.

309. Expressio secunda $\frac{x^3}{(1-x)(1-xx)}$, dat hanc Seriem

$$x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \&c.,$$

in qua cujusvis termini coëfficiens indicat quot modis Exponens ipsius x in duas partes inæquales dispertiri possit. Sic terminus $4x^9$ indicat, numerum 9 quatuor modis in duas partes inæquales secari posse. Quod si hanc Seriem per x^3 dividamus, prodibit Series, quam præbet ista fractio $\frac{1}{(1-x)(1-x^2)}$, quæ erit

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \&c.,$$

cujus terminus generalis sit $= Nx^n$; atque ex genesi hujus Seriei intelligitur coëfficientem N indicare, quot variis modis Exponens n ex numeris 1 & 2 per additionem nasci queat. Cum igitur prioris Seriei terminus generalis sit $= Nx^{n+3}$, deducitur hinc istud theorema.

Quot variis modis numerus n *per additionem ex numeris* 1 & 2 *produci potest, totidem variis modis numerus* n + 3 *in duas partes inæquales secari poterit.*

310. Expressio tertia $\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$ in Seriem evoluta dabit

$$x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + \&c.,$$

in qua cujusvis termini coëfficiens indicat quot variis modis Exponens Potestatis x adjunctæ in tres partes inæquales dispertiri

tiri possit. Quod si autem hæc fractio $\frac{1}{(1-x)(1-x^2)(1-x^3)}$ evolvatur, prodibit hæc Series CAP. XVI.

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + \&c.,$$

cujus terminus generalis si ponatur $= Nx^n$, coëfficiens N indicabit quot variis modis numerus n ex numeris 1, 2, 3, per additionem produci possit. Cum igitur prioris Seriei terminus generalis sit Nx^{n+6}, sequetur hinc istud theorema.

Quot variis modis numerus n *per additionem ex numeris* 1, 2, 3, *produci potest, totidem variis modis numerus* n + 6 *in tres partes inæquales secari poterit.*

311. Expressio quarta $\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ in Seriem recurrentem evoluta dabit

$$x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + \&c.,$$

in qua cujusvis termini coëfficiens indicabit quot variis modis Exponens Potestatis x adjunctæ in quatuor partes inæquales dispertiri possit. Quod si autem hæc expressio $\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ evolvatur, prodibit superior Series per x^{10} divisa, nempe

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + \&c.,$$

cujus terminum generalem ponamus $= Nx^n$; atque hinc patebit coëfficientem N indicare, quot variis modis numerus n per additionem oriri possit ex his quatuor numeris 1, 2, 3, 4. Cum igitur prioris Seriei terminus generalis futurus sit $= Nx^{n+10}$, deducitur hoc theorema.

 Quot

LIB. I. *Quot variis modis numerus* n *per additionem produci potest ex numeris* 1, 2, 3, 4, *totidem variis modis numerus* n + 10 *in quatuor partes inæquales secari poterit.*

312. Generaliter ergo, si hæc expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^m)}$$

in Seriem evolvatur, ejusque terminus generalis fuerit $= Nx^n$, coëfficiens N indicabit, quot variis modis numerus n per additionem produci possit ex his numeris 1, 2, 3, 4 .... m. Quod si autem hæc expressio

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^m)}$$

in Seriem evolvatur, erit ejus terminus generalis $= Nx^{n+\frac{m(m+1)}{2}}$: atque hic coëfficiens N indicat quot variis modis numerus $n+\frac{m(m+1)}{1.\ 2}$ in m partes inæquales secari possit, unde hoc habetur theorema.

Quot variis modis numerus n *per additionem produci potest ex numeris* 1, 2, 3, 4 .... m, *totidem modis numerus* $\mathrm{n}+\frac{\mathrm{m}(\mathrm{m}+1)}{1.\ 2}$ *in* m *partes inæquales secari poterit.*

313. Ex posita partitione numerorum in partes inæquales, perpendamus quoque partitionem in partes, ubi æqualitas partium non excluditur; quæ partitio ex hac expressione originem habet

$$Z = \frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\ \&c.}$$

Ponamus evolutione per divisionem instituta prodire

Z =

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \&c..$$ CAP. XVI.

Perſpicuum autem eſt, ſi loco z ponatur xz, prodire

$$\frac{1}{(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\&c.} = (1-xz)Z.$$

Facta ergo in Serie evoluta eadem mutatione, fiet

$$(1-xz)Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c..$$

Multiplicetur ergo ſuperior Series pariter per $(1-xz)$, eritque

$$\begin{aligned}(1-xz)Z = 1 &+ Pz + Qz^2 + Rz^3 + Sz^4 + \&c. \\ &- xz - Pxz^2 - Qxz^3 - Rxz^4 - \&c..\end{aligned}$$

Comparatione ergo inſtituta orietur

$$P = \frac{x}{1-x};\ Q = \frac{Px}{1-x^2};\ R = \frac{Qx}{1-x^3};\ S = \frac{Rx}{1-x^4}\ \&c.,$$

unde pro P, Q, R, S, &c., ſequentes valores proveniunt.

$$P = \frac{x}{1-x}$$

$$Q = \frac{x^2}{(1-x)(1-x^2)}$$

$$R = \frac{x^3}{(1-x)(1-x^2)(1-x^3)}$$

$$S = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

&c.

314. Expreſſiones iſtæ a ſuperioribus aliter non diſcrepant, niſi quod numeratores hic minores habeant Exponentes quam caſu præcedente. Atque hanc ob rem Series, quæ per evolutionem naſcuntur, ratione coëfficientium omnino convenient, quæ convenientia jam ex comparatione (§. §. 300. & 304.) perſpi-

LIB. I. perſpicitur, nunc vero demum ejus ratio intelligitur. Hinc ergo omnino ſimilia theoremata conſequentur, quæ ſunt.

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, *totidem modis numerus* $n+2$ *in duas partes diſpertiri poterit.*

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, *totidem modis numerus* $n+3$ *in tres partes diſpertiri poterit.*

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, 4, *totidem modis numerus* $n+4$ *in quatuor partes diſpertiri poterit.*

Atque generaliter habebitur hoc theorema:

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, ... m, *totidem modis numerus* $n+m$ *in* m *partes diſpertiri poterit.*

315. Sive ergo quæratur quot modis datus numerus in *m* partes inæquales, ſive in *m* partes, æqualibus non excluſis, diſpertiri poſſit, utraque quæſtio reſolvetur ſi cognoſcatur quot modis quiſque numerus per additionem produci poſſit ex numeris 1, 2, 3, 4 .... *m*, quemadmodum hoc patebit ex ſequentibus theorematis, quæ ex ſuperioribus ſunt derivata.

Numerus n *tot modis in* m *partes inæquales diſpertiri poteſt, quot modis numerus* $n - \frac{m(m+1)}{2}$ *per additionem produci poteſt ex numeris* 1, 2, 3, 4, ... m.

Numerus n, *tot modis in* m *partes ſive æquales ſive inæquales diſpertiri poteſt quot modis numerus* $n - m$ *per additionem produci poteſt ex numeris* 1, 2, 3, ... m.

Hinc porro ſequuntur hæc theoremata.

Numerus n *totidem modis in* m *partes inæquales ſecari poteſt, quot modis numerus* $n - \frac{m(m-1)}{2}$ *in* m *partes, ſive æquales ſive inæquales, diſpertitur.*

Numerus n *totidem modis in* m *partes, ſive inæquales ſive æquales,*

ſecari

secari potest, quot modis numerus $n + \frac{m(m-1)}{2}$ *in* m *partes inæquales dispertiri potest.*

316. Per formationem autem Serierum recurrentium inveniri poterit, quot variis modis datus numerus n per additionem produci possit ex numeris $1, 2, 3, \ldots\ldots m$. Ad hoc enim inveniendum evolvi debebit fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

atque Series recurrens continuari debebit usque ad terminum Nx^n, cujus coëfficiens N indicabit, quot modis numerus n per additionem produci possit ex numeris $1, 2, 3, 4, \ldots\ldots m$. At vero hic solvendi modus non parum habebit difficultatis, si numeri m & n sint modice magni; scala enim relationis, quam præbet denominator per multiplicationem evolutus, ex pluribus terminis constat, unde operosum erit Seriem ad plures terminos continuare.

317. Hæc autem disquisitio minus erit molesta, si casus simpliciores primum expediantur, ex his enim facile erit ad casus magis compositos progredi. Sit Seriei, quæ ex hac fractione oritur,

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

terminus generalis $= Nx^n$; at Seriei ex hac forma

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

ortæ terminus generalis sit Mx^n, ubi coëfficiens M indicabit quot variis modis numerus $n-m$ per additionem produci

LIB. I. possit ex numeris 1, 2, 3, ... m. Subtrahatur posterior expressio a priori, ac remanebit

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^{m-1})}$$

atque manifestum est Seriei hinc ortæ terminum generalem futurum esse $(N-M)x^n$; quare coëfficiens $N-M$ indicabit quot variis modis numerus n per additionem produci possit ex numeris 1, 2, 3, ... $(m-1)$.

318. Hinc ergo sequentem regulam nanciscimur.

Sit L numerus modorum, quibus numerus n per additionem produci potest ex numeris 1, 2, 3, ... $(m-1)$.

Sit M numerus modorum, quibus numerus $n-m$ per additionem produci potest ex numeris 1, 2, 3, ... m.

Sitque N numerus modorum, quibus numerus n per additionem produci potest ex numeris 1, 2, 3, ... m.

His positis, erit, ut vidimus, $L=N-M$; ideoque $N=L+M$. Quod si ergo jam invenerimus quot variis modis numeri n & $n-m$ per additionem produci queant, ille ex numeris 1, 2, 3, ... $(m-1)$ hic vero ex numeris 1, 2, 3, ... m; hinc addendo cognoscemus, quot variis modis numerus n per additionem produci queat ex numeris 1, 2, 3, ... m. Ope hujus theorematis a casibus simplicioribus, qui nihil habent difficultatis, continuo ad magis compositos progredi licebit, hocque modo tabula hic annexa est computata, * cujus usus ita se habet.

Si quæratur quot variis modis numerus 50 in 7 partes inæquales dispertiri possit; sumatur in prima columna verticali numerus $50-\frac{7\cdot 8}{2}=22$, in horizontali autem suprema numerus romanus VII; atque numerus in angulo positus 522 indicabit modorum numerum quæsitum.

Sin autem quæratur, quot variis modis numerus 50 in 7 partes, sive æquales sive inæquales, dispertiri possit, in prima columna

* Vide Tab. pag. 275.

columna verticali ſumatur numerus $50 - 7 = 43$, cui in columna 7ma reſpondebit numerus quæſitus 8946. CAP. XVI.

319. Series hujus tabulæ verticales, etſi ſunt recurrentes, tamen ingentem habent connexionem cum numeris naturalibus, trigonalibus, pyramidalibus, & ſequentibus, quam paucis exponere operæ pretium erit. Quoniam enim ex fractione $\frac{1}{(1-x)(1-xx)}$ oritur Series $1+x+2x^2+2x^3+3x^4+3x^5+$ &c., ac proinde ex fractione $\frac{x}{(1-x)(1-xx)}$ hæc $x+x^2+2x^3+2x^4+3x^5+3x^6+$ &c.. Si duæ hæ Series addantur, naſcitur iſta

$$1+2x+3x^2+4x^3\ 5x^4+6x^5+7x^6+\text{\&c.},$$

quæ per diviſionem oritur ex fractione $\frac{1+x}{(1-x)(1-xx)} = \frac{1}{(1-x)^2}$; unde patet Seriei poſtremæ terminos numericos Seriem numerorum naturalium conſtituere. Hinc ex Serie tabulæ ſecunda addendo binos terminos proveniet Series numerorum naturalium, poſito $x=1$.

$$\begin{array}{l} 1+1+2+2+3+3+4+4+5+5+6+6+\text{\&c.} \\ 1+2+3+4+5+6+7+8+9+10+11+12+\text{\&c.} \end{array}$$

Viciſſim ergo ex Serie numerorum naturalium ſuperior invenitur, ſubtrahendo quemque terminum Seriei ſuperioris a termino inferioris ſequente.

320. Series verticalis tertia oritur ex fractione $\frac{1}{(1-x)(1-xx)(1-x^3)}$. Cum autem ſit $\frac{1}{(1-x)^3} = \frac{(1+x)(1+x+xx)}{(1-x)(1-xx)(1-x^3)}$, manifeſtum eſt, ſi primo Seriei illius terni termini addantur, tum bini hujus novæ Seriei denuo addantur, prodire debere numeros trigonales, id quod ex ſchemate ſequente apparebit

LIB. I. 1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + 14 + 16 + 19 &c.
1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + 36 + 42 + 49 &c.
1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + 66 + 78 + 91 &c.

Viciſſim autem apparet quomodo ex Serie trigonalium erui debeat Series ſuperior.

321. Simili modo, quia Series quarta oritur ex fractione $\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)}$, erit $\frac{(1+x)(1+x+xx)(1+x+xx+x^3)}{(1-x)(1-xx)(1-x^3)(1-x^4)}$ $= \frac{1}{(1-x)^4}$. Si in Serie quarta primum quaterni termini addantur, tum in Serie reſultante terni, denique in hac bini, prodibit Series numerorum pyramidalium uti ex ſequenti calculo videre licet.

1 + 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + 23 + 27 + &c.
1 + 2 + 4 + 7 + 11 + 16 + 23 + 31 + 41 + 53 + 67 + 83 + &c.
1 + 3 + 7 + 13 + 22 + 34 + 50 + 70 + 95 + 125 + 161 + 203 + &c.
1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 + 220 + 286 + 364 + &c.

Simili autem modo Series quinta deducet ad numeros pyramidales ſecundi ordinis, ſexta ad tertii ordinis, & ita porro.

322. Viciſſim igitur ex numeris figuratis illæ ipſæ Series, quæ in tabulis occurrunt, formari poterunt, per operationes, quæ ex inſpectione calculi ſequentis ſponte elucebunt.

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + &c.
1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + &c. II

1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + &c.
1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + &c.
1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + &c. III

1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 + 220 + &c.
1 + 3 + 7 + 13 + 22 + 34 + 50 + 70 + 95 + 125 + &c.
1 + 2 + 4 + 7 + 11 + 16 + 23 + 31 + 41 + 53 + &c.
1 + 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + &c. IV

1 + 5 + 15 + 35 + 70 + 126 + 210 + 330 + 495 + 715 + &c.
1 + 4 + 11 + 24 + 46 + 80 + 130 + 200 + 295 + 420 + &c.
1 + 3 + 7 + 14 + 25 + 41 + 64 + 95 + 136 + 189 + &c.
1 + 2 + 4 + 7 + 12 + 18 + 27 + 38 + 53 + 71 + &c.
1 + 1 + 2 + 3 + 5 + 7 + 10 + 13 + 18 + 23 + &c. V

&c.

In his ordinibus primæ Series ſunt numeri figurati, unde ſubtrahendo quemvis terminum Seriei ſecundæ a termino primæ ſequente formatur Series ſecunda. Tum Seriei tertiæ bini termini conjunctim ſubtrahantur a termino ſequente Seriei ſecundæ, ſicque oritur Series tertia; hocque modo ſubtrahendo ulterius ſummam trium, quatuor, & ita porro terminorum a termino ſuperioris Seriei ſequente, formabuntur reliquæ Series donec perveniatur ad Seriem, quæ incipit ab 1 + 1 + 2 &c., hæcque erit Series in tabula exhibita.

323. Series verticales tabulæ omnes ſimiliter incipiunt, continuoque plures habent terminos communes; ex quo intelligitur in infinitum has Series inter ſe fore congruentes. Prodibit autem Series, quæ oritur ex hac fractione

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\,\&c.},$$

quæ cum ſit recurrens, primum denominator ſpectari debet, ut

hinc ſcala relationis habeatur. Quod ſi autem Factores denominatoris continuo in ſe multiplicentur, prodibit

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+\&c.,$$

quæ Series ſi attentius conſideretur, aliæ Poteſtates ipſius x adeſſe non deprehenduntur, niſi quarum Exponentes contineantur in hac formula $\frac{3nn\pm n}{2}$; atque, ſi n ſit numerus impar, Poteſtates erunt negativæ; affirmativæ autem ſi n fuerit numerus par.

324. Cum igitur ſcala relationis ſit

$$+1, +1, 0, 0, -1, 0, -1, 0, 0, 0, 0, +1, 0, 0, +1, 0, 0, \&c.,$$

Series recurrens ex evolutione fractionis

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6((1-x^7)\&c.,}$$

oriunda erit hæc

$$1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+30x^9+42x^{10}+56x^{11}+77x^{12}+101x^{13}+135x^{14}+176x^{15}+231x^{16}+297x^{17}+385x^{18}+490x^{19}+627x^{20}+792x^{21}+1002x^{22}+1250x^{23}+1570x^{24}\ \&c..$$

In hac ergo Serie coëfficiens quiſque indicat, quot variis modis Exponens ipſius x per additionem ex numeris integris oriri queat. Sic numerus 7 quindecim modis per additionem oriri poteſt.

$7=7$	$7=4+2+1$	$7=3+1+1+1+1$
$7=6+1$	$7=4+1+1+1$	$7=2+2+2+1$
$7=5+2$	$7=3+3+1$	$7=2+2+1+1+1$
$7=5+1+1$	$7=3+2+2$	$7=2+1+1+1+1+1$
$7=4+3$	$7=3+2+1+1$	$7=1+1+1+1+1+1+1$

325. Quod

325. Quod ſi autem hoc productum CAP. XVI.

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\ \&c.,$$

evolvatur, ſequens prodibit Series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+\&c.,$$

in qua quiſque coëfficiens indicat, quot variis modis Exponens ipſius x per additionem numerorum inæqualium oriri poſſit, Sic numerus 9 octo variis modis per additionem ex numeris inæqualibus formari poteſt.

$9=9$	$9=6+2+1$
$9=8+1$	$9=5+4$
$9=7+2$	$9=5+3+1$
$9=6+3$	$9=4+3+2$

326. Ut comparationem inter has formas inſtituamus, ſit

$$P=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\ \&c.,$$

&

$$Q=(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\ \&c.,$$

erit

$$PQ=(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\ \&c.,$$

qui Factores cum omnes in P contineantur, dividatur P per PQ, erit $\frac{1}{Q}=(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\ \&c.,$

ideoque

$$Q=\frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\ \&c.},$$

quæ fractio ſi evolvatur, prodibit Series, in qua quiſque coëfficiens indicabit, quot variis modis Exponens ipſius x, per additionem ex numeris imparibus produci poſſit. Cum igitur hæc expreſſio æqualis ſit illi, quam in §. præcedente contemplati ſumus, ſequitur hinc iſtud theorema.

Quot

LIB. I. *Quot modis datus numerus per additionem formari potest ex omnibus numeris integris inter se inæqualibus; totidem modis idem numerus formari poterit per additionem ex numeris tantum imparibus, sive æqualibus sive inæqualibus.*

327. Cum igitur, ut ante vidimus, sit

$$P = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \text{\&c.},$$ erit, scribendo xx loco x,

$$PQ = 1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + x^{44} + x^{52} - \text{\&c.},$$

Quocirca erit hanc per illam dividendo

$$Q = \frac{1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + \text{\&c.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{\&c.}}.$$

Erit ergo Series Q pariter recurrens, atque ex Serie $\frac{1}{P}$ oritur, hanc per $1 - x^2 - x^4 + x^{10} + x^{14} - x^{24}$ &c., multiplicando. Nempe, cum sit ex (324), $\frac{1}{P} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{\&c.}$, si is multiplicetur per

$$1 - x^2 - x^4 + x^{10} + x^{14} - \text{\&c.},$$

fiet

$$\begin{aligned} 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{\&c.} \\ - x^2 - x^3 - 2x^4 - 3x^5 - 5x^6 - 7x^7 - 11x^8 - 15x^9 - \text{\&c.} \\ - x^4 - x^5 - 2x^6 - 3x^7 - 5x^8 - 7x^9 - \text{\&c.} \end{aligned}$$

aut

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \text{\&c.}$$

$= Q$ Hinc ergo, si formatio numerorum per additionem numerorum, sive æqualium sive inæqualium constet, deducetur formatio numerorum per additionem numerorum inæqualium, hincque porro formatio numerorum per additionem numerorum imparium tantum.

328. Restant in hoc genere casus quidam memorabiles, quorum evolutio non omni utilitate carebit in numerorum natura cognoscenda. Consideretur nempe hæc expressio

$(1 + x)$

$$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\ \&c.,$$ CAP. XVI.

in qua Exponentes ipſius x in ratione dupla progrediuntur. Hæc expreſſio ſi evolvatur, reperietur quidem hæc Series

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\&c.;$$

quoniam vero dubium eſſe poteſt, utrum hæc Series in infinitum hac lege geometrica progrediatur, hanc ipſam Seriem inveſtigemus. Sit igitur

$$P=(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\ \&c.,$$

ac ponatur Series per evolutionem oriunda

$$P=1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\zeta x^6+\eta x^7+\theta x^8+\&c.,$$

Patet autem ſi loco x ſcribatur xx, tum prodire productum $(1+xx)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\&c.=\frac{P}{1+x}$: facta ergo in Serie eadem ſubſtitutione erit

$$\frac{P}{1+x}=1+\alpha x^2+\beta x^4+\gamma x^6+\delta x^8+\varepsilon x^{10}+\zeta x^{12}+\&c.,$$

multiplicetur ergo per $1+x$, eritque

$$P=1+x+\alpha x^2+\alpha x^3+\beta x^4+\beta x^5+\gamma x^6+\gamma x^7+\delta x^8+\delta x^9+\&c.,$$

qui valor ipſius P ſi cum ſuperiori comparetur, habebitur

$$\alpha=1;\ \beta=\alpha;\ \gamma=\alpha;\ \delta=\beta;\ \varepsilon=\beta;\ \zeta=\gamma;\ \eta=\gamma;\ \&c.,$$

erunt ergo omnes coëfficientes $=1$, ideoque productum propoſitum P evolutum dabit Seriem geometricam

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+\&c.,$$

329. Cum igitur hic omnes ipſius x Poteſtates, ſingulæque ſemel occurrant, ex forma producti $(1+x)(1+x^2)(1+x^4)$ &c., ſequitur, omnem numerum integrum ex terminis progreſſionis

 geo-

LIB. I. geometricæ duplæ 1, 2, 4, 8, 16, 32, &c., diversis per additionem formari posse, hocque unico modo. Nota est hæc proprietas in praxi ponderandi, si enim habeantur pondera 1, 2, 4, 8, 16, 32, &c., librarum; his solis ponderibus omnia onera ponderari poterunt, nisi partes libræ requirant. Sic his decem ponderibus, nempe 1 ℔, 2 ℔, 4 ℔, 8 ℔, 16 ℔, 32 ℔, 64 ℔, 128 ℔, 256 ℔, 512 ℔, omnia pondera usque ad 1024 ℔, librari possunt, & si unum pondus 1024 ℔, addatur omnibus oneribus usque ad 2048 ℔, ponderandis sufficient.

330. Ostendi autem insuper solet in praxi ponderandi paucioribus ponderibus, quæ scilicet in ratione geometrica tripla progrediantur, nempe 1, 3, 9, 27, 81, &c, librarum pariter omnia onera ponderari posse, nisi opus sit fractionibus. In hac autem praxi pondera non solum uni lanci, sed ambabus, uti necessitas exigit, imponi debent. Nititur ergo ista praxis hoc fundamento, quod ex terminis progressionis geometricæ triplæ 1, 3, 9, 27, 81, &c., diversis semper sumendis per additionem ac subtractionem omnes omnino numeri produci queant; erit scilicet.

$1 = 1$	$5 = 9 - 3 - 1$	$9 = 9$
$2 = 3 - 1$	$6 = 9 - 3$	$10 = 9 + 1$
$3 = 3$	$7 = 9 - 3 + 1$	$11 = 9 + 3 - 1$
$4 = 3 + 1$	$8 = 9 - 1$	$12 = 9 + 3$

&c.

331. Ad hanc veritatem ostendendam considero hoc productum infinitum

$$(x^{-1}+1+x^{1})(x^{-3}+1+x^{3})(x^{-9}+1+x^{9})(x^{-27}+1+x^{27})\ \&c.$$
$$= P,$$

quod evolutum alias non dabit Potestates ipsius x, nisi quarum Exponentes formari possint ex numeris 1, 3, 9, 27, 81, &c., sive

TABULA

ad paginam 275 Tom. I.

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3	3	3
4	1	3	4	5	5	5	5	5	5	5	5
5	1	3	5	6	7	7	7	7	7	7	7
6	1	4	7	9	10	11	11	11	11	11	11
7	1	4	8	11	13	14	15	15	15	15	15
8	1	5	10	15	18	20	21	22	22	22	22
9	1	5	12	18	23	26	28	29	30	30	30
10	1	6	14	23	30	35	38	40	41	42	42
11	1	6	16	27	37	44	49	52	54	55	56
12	1	7	19	34	47	58	65	70	73	75	76
13	1	7	21	39	57	71	82	89	94	97	99
14	1	8	24	47	70	90	105	116	123	128	131
15	1	8	27	54	84	110	131	146	157	164	169
16	1	9	30	64	101	136	164	186	201	212	219
17	1	9	33	72	119	163	201	230	252	267	278
18	1	10	37	84	141	199	248	288	318	340	355
19	1	10	40	94	164	235	300	352	393	423	445
20	1	11	44	108	192	282	364	434	488	530	560
21	1	11	48	120	221	331	436	525	598	653	695
22	1	12	52	136	255	391	522	638	732	807	863
23	1	12	56	150	291	454	618	764	887	984	1060
24	1	13	61	169	333	532	733	919	1076	1204	1303
25	1	13	65	185	377	612	860	1090	1291	1455	1586
26	1	14	70	206	427	709	1009	1297	1549	1761	1930
27	1	14	75	225	480	811	1175	1527	1845	2112	2331
28	1	15	80	249	540	931	1367	1801	2194	2534	2812
29	1	15	85	270	603	1057	1579	2104	2592	3015	3370
30	1	16	91	297	674	1206	1824	2462	3060	3590	4035
31	1	16	96	321	748	1360	2093	2857	3589	4242	4802
32	1	17	102	351	831	1540	2400	3319	4206	5013	5788
33	1	17	108	378	918	1729	2738	3828	4904	5888	6751
34	1	18	114	411	1014	1945	3120	4417	5708	6912	7972
35	1	18	120	441	1115	2172	3539	5066	6615	8070	9373
36	1	19	127	478	1226	2432	4011	5812	7657	9418	11004
37	1	19	133	511	1342	2702	4526	6630	8824	10936	12866
38	1	20	140	551	1469	3009	5102	7564	10156	12690	15021
39	1	20	147	588	1602	3331	5731	8588	11648	14663	17475
40	1	21	154	632	1747	3692	6430	9749	13338	16928	20298
41	1	21	161	672	1898	4070	7190	11018	15224	19466	23501
42	1	22	169	720	2062	4494	8033	12450	17354	22367	27169
43	1	22	176	764	2233	4935	8946	14012	19720	25608	31316
44	1	23	184	816	2418	5427	9953	15765	22380	29292	36043
45	1	23	192	864	2611	5942	11044	17674	25331	33401	41373
46	1	24	200	920	2818	6510	12241	19805	28629	38047	47420
47	1	24	208	972	3034	7104	13534	22122	32278	43214	54218
48	1	25	217	1033	3266	7760	14950	24699	36347	49037	61903
49	1	25	225	1089	3507	8442	16475	27493	40831	55494	70515
50	1	26	234	1154	3765	9192	18138	30588	45812	62740	80215
51	1	26	243	1215	4033	9975	19928	33940	51294	70760	91058
52	1	27	252	1285	4319	10829	21873	37638	57358	79725	103226
53	1	27	261	1350	4616	11720	23961	41635	64015	89623	116792
54	1	28	271	1425	4932	12692	26226	46031	71362	100654	131970
55	1	28	280	1495	5260	13702	28652	50774	79403	112804	148847
56	1	29	290	1575	5608	14800	31275	55974	88252	126299	167672
57	1	29	300	1650	5969	15944	34082	61575	97922	141136	188556
58	1	30	310	1735	6351	17180	37108	67696	108527	157564	211782
59	1	30	320	1815	6747	18467	40340	74280	120092	175586	237489
60	1	31	331	1906	7166	19858	43819	81457	132751	195491	266006
61	1	31	341	1991	7599	21301	47527	89162	146520	217280	297495
62	1	32	352	2087	8056	22856	51508	97539	161554	241279	332337
63	1	32	363	2178	8529	24473	55748	106522	177884	267507	370733
64	1	33	374	2280	9027	26207	60289	116263	195666	296320	413112
65	1	33	385	2376	9542	28009	65117	126692	214944	327748	459718
66	1	34	397	2484	10083	29941	70281	137977	235899	362198	511045
67	1	34	408	2586	10642	31943	75762	150042	258569	399705	567377
68	1	35	420	2700	11229	34085	81612	163069	283161	440725	629281
69	1	35	432	2808	11835	36308	87816	176978	309729	485315	697097

ſive addendo ſive ſubtrahendo: num vero omnes Poteſtates prodeant, ſingulæque ſemel, ſic exploro. Sit Cap. XVI.

$$P = \&c. + cx^{-3} + bx^{-2} + ax^{-1} + 1 + \alpha x^1 + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.,$$

manifeſtum vero eſt, ſi x^3 loco x ſcribatur, tum prodire

$$\frac{P}{x^{-1} + 1 + x^1} = bx^{-6} + ax^{-3} + 1 + \alpha x^3 + \beta x^6 + \gamma x^9 + \&c..$$

Hinc igitur reperitur $P = \&c.$

$$+ ax^{-4} + ax^{-3} + ax^{-2} + x^{-1} + 1 + x + \alpha x^2 + \alpha x^3 + \alpha x^4 + \beta x^5 + \beta x^6 + \beta x^7 + \&c.,$$

quæ expreſſio cum aſſumta comparata dabit

$$\alpha = 1;\ \beta = \alpha;\ \gamma = \alpha;\ \delta = \alpha;\ \varepsilon = \beta;\ \xi = \beta;\ \&c., \ \&$$
$$a = 1,\ b = a,\ c = a,\ d = a,\ e = b,\ \&c..$$

Hinc itaque erit

$$P = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c.$$
$$+ x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + \&c.,$$

unde patet omnes ipſius x Poteſtates, tam affirmativas quam negativas, hic occurrere, atque adeo omnes numeros ex terminis progreſſionis geometricæ triplæ, vel addendo vel ſubtrahendo, formari poſſe; & unumquemque numerum unico tantum modo.

LIB. I.

CAPUT XVII.

De usu Serierum recurrentium in radicibus æquationum indagandis.

332. INdicavit *Vir Celeb. Daniel* BERNOULLI insignem usum Serierum recurrentium in investigandis radicibus æquationum cujusvis gradus, in *Comment. Acad. Petropol. Tomo III.*, ubi ostendit, quemadmodum cujusque æquationis algebraicæ, quotcunque fuerit dimensionum, valores radicum veris proximi ope Serierum recurrentium assignari queant. Quæ inventio, cum sæpenumero maximam afferat utilitatem, eam hic diligentius explicare constitui, ut intelligatur, quibus casibus adhiberi possit. Interdum enim præter expectationem evenit, ut nulla æquationis radix ope hujus methodi cognosci queat. Quocirca, ut vis hujus methodi clarius perspiciatur, ex proprietatibus Serierum recurrentium totum fundamentum, quo nititur, contemplemur.

333. Quoniam omnis Series recurrens ex evolutione cujusdam fractionis rationalis oritur, sit ista fractio

$$= \frac{a + bz + cz^2 + dz^3 + ez^4 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.},$$

unde oriatur sequens Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes A, B, C, D, &c., ita determinantur ut sit

$A = a$

$A = a$ Cap. XVII.
$B = \alpha A + b$
$C = \alpha B + \beta A + c$
$D = \alpha C + \beta B + \gamma A + d$
$E = \alpha D + \beta C + \gamma B + \delta A + e$
&c.

Terminus autem generalis, ſeu coëfficiens Poteſtatis z^n, invenitur ex reſolutione fractionis propoſitæ in fractiones ſimplices, quarum denominatores ſint Factores denominatoris $1 - \alpha z - \beta zz - \gamma z^3 -$ &c., uti (*Cap. XIII.*) eſt oſtenſum.

334. Forma autem termini generalis potiſſimum pendet ab indole Factorum ſimplicium denominatoris, utrum ſint reales an imaginarii, & utrum ſint inter ſe inæquales & eorum bini pluresve æquales. Quos varios caſus ut ordine percurramus, ponamus primum omnes denominatoris Factores ſimplices cum reales eſſe tum inter ſe inæquales. Sint ergo Factores ſimplices denominatoris omnes $(1-pz)(1-qz)(1-rz)(1-sz)$ &c., ex quibus fractio propoſita in ſequentes fractiones ſimplices reſolvatur $\frac{A}{1-pz} + \frac{B}{1-qz} + \frac{C}{1-rz} + \frac{D}{1-sz} +$ &c.. Quibus cognitis erit Seriei recurrentis terminus generalis $= z^n (Ap^n + Bq^n + Cr^n + Ds^n +$ &c.$)$, quem ſtatuamus $= Pz^n$; ſit ſcilicet P coëfficiens Poteſtatis z^n, ſequentiumque Q, R, &c., ita ut Series recurrens fiat

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \text{\&c.}$$

335. Ponamus jam n eſſe numerum maximum, ſeu Seriem recurrentem ad plurimos terminos eſſe continuatam; quoniam numerorum inæqualium Poteſtates eo magis fiunt inæquales, quo fuerint altiores; tanta erit diverſitas in Poteſtatibus

 Ap^n

Ap^n, Bq^n, Cr^n, &c., ut ea, quæ oritur ex maximo numerorum p, q, r, &c., reliquas magnitudine longe ſuperet, præ eaque reliquæ penitus evaneſcant, ſi n fuerit numerus plane infinite magnus. Cum igitur numeri p, q, r, &c., ſint inter ſe inæquales, ponamus inter eos p eſſe maximum; ac propterea, ſi n ſit numerus infinitus, fiet $P = Ap^n$; ſin autem n ſit numerus vehementer magnus erit tantum proxime $P = Ap^n$. Simili vero modo erit $Q = Ap^{n+1}$, ideoque $\frac{Q}{P} = p$. Unde patet, ſi Series recurrens jam longe fuerit producta, coëfficientem cujuſque termini per præcedentem diviſum proxime eſſe exhibiturum valorem maximæ litteræ p.

336. Si igitur in fractione propoſita

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

denominator habeat omnes Factores ſimplices reales & inter ſe inæquales, ex Serie recurrente inde orta cognoſci poterit unus Factor ſimplex, is ſcilicet $1 - pz$, in quo littera p omnium maximum habet valorem. Neque in hoc negotio coëfficientes numeratoris a, b, c, d. &c., in computum ingrediuntur, ſed quicunque ii ſtatuantur, tamen denique idem verus valor litteræ maximæ p invenitur. Verus quidem valor ipſius p tum demum innoteſcit, quando Series in infinitum fuerit continuata; interim tamen ſi jam plures ejus termini fuerint formati, eo propius valor ipſius p cognoſcetur, quo major fuerit terminorum numerus, & quo magis littera iſta p excedat reliquas q, r, s, &c.: perinde vero eſt utrum hæc maxima littera p fuerit ſigno $+$ an ſigno $-$ affecta, quoniam ejus Poteſtates æque increſcunt.

337. Quemadmodum nunc hæc inveſtigatio ad inventionem radicum æquationis cujuſvis algebraicæ accommodari poſſit,

fit, ſatis eſt perſpicuum. Ex Factoribus enim denominatoris $1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 -$ &c., cognitis facile aſſignantur radices æquationis hujus CAP. XVII.

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c. = 0,$$

ita ut, ſi Factor fuerit $1 - pz$, hujus æquationis radix una futura ſit $z = \frac{1}{p}$. Cum igitur ex Serie recurrente reperiatur maximus numerus p, indidem obtinebitur minima radix æquationis $1 - \alpha z - \beta z^2 - \gamma z^3 -$ &c. $= 0$. Vel, ſi ponatur $z = \frac{1}{x}$ ut prodeat hæc æquatio

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

ejuſdem methodi ope eruitur maxima hujus æquationis radix $x = p$.

338. Si igitur proponatur æquatio hæc

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

quæ omnes radices habeat reales & inter ſe inæquales, harum radicum maxima ſequenti modo reperietur. Formetur ex coëfficientibus hujus æquationis fractio

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

Hincque formetur Series recurrens, aſſumendo pro arbitrio numeratorem, ſeu, quod eodem redit, aſſumendo pro libitu terminos initiales; quæ ſit

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots\ldots + Pz^n + Qz^{n+1}$$

dabitque fractio $\frac{Q}{P}$ valorem radicis maximæ x pro æquatione propoſita, eo propius, quo major fuerit numerus n.

EXEM-

LIB. I.

EXEMPLUM I.

Sit proposita ista æquatio $xx - 3x - 1 = 0$, *cujus maximam radicem inveniri oporteat.*

Formetur fractio $\frac{a + bz}{1 - 3z - zz}$, unde positis duobus primis terminis $1, 2$, orietur ista Series recurrens

$$1, 2, 7, 23, 76, 251, 829, 2738, \&c.,$$

erit ergo $\frac{2738}{829}$ proxime æqualis radici æquationis propositæ maximæ. Valor autem hujus fractionis in partibus decimalibus expressus est

$$3, 3027744$$

æquationis vero radix maxima est $= \frac{3 + \sqrt{13}}{2} =$

$$3, 3027756,$$

quæ inventam superat tantum una parte millionesima. Ceterum notandum est fractiones $\frac{Q}{P}$ alternatim vera radice esse majores & minores.

EXEMPLUM II.

Proposita sit ista æquatio $3x - 4x^3 = \frac{1}{2}$ *cujus radices exhibent Sinus trium Arcuum, quorum triplorum Sinus est* $= \frac{1}{2}$.

Æquatione perducta ad hanc formam $0 = 1 - 6x * + 8x^3$, quæratur hujus, ut in numeris integris maneamus, radix minima, ita ut non opus sit pro x ponere $\frac{1}{z}$. Formetur ergo hæc fractio

$$\frac{a + bx + cxx}{1 - 6x * + 8x^3}$$

ex qua sumendis pro lubitu tribus terminis initialibus $0, 0, 1$, quia

quia hoc modo calculus facillime expeditur, orietur hæc Series recurrens, omittendis poteſtatibus ipſius x quia tantum coëfficientibus opus eſt, CAP. XVII.

0; 0; 1; 6; 36; 208; 1200; 6912; 39808; 229248.

Erit ergo proxime æquationis radix minima $= \frac{39808}{229248} = \frac{311}{1791} = 0,1736515$, quæ propterea eſſe deberet Sinus anguli $10°$; hic autem ex tabulis eſt 0,1736482, quem ſuperat radix inventa parte $\frac{33}{10000000}$. Facilius autem hæc eadem radix inveniri poteſt ponendo $x = \frac{1}{2}y$, ut prodeat æquatio $1 - 3y * + y^3 = 0$, ex qua ſimili modo tractata oritur Series

0, 0, 1, 3, 9, 26, 75, 216, 622, 1791, 5157 &c.,

erit ergo proxime æquationis radix minima $y = \frac{1791}{5157} = \frac{199}{573} = 0,3472949$, unde fit $x = \frac{1}{2}y = 0,1736479$, qui valor decies propius accedit quam præcedens.

EXEMPLUM III.

Si deſideretur ejuſdem æquationis propoſitæ $0 = 1 - 6x * + 8x^3$, *radix maxima.*

Ponatur $x = \frac{y}{2}$, eritque $y^3 * - 3y + 1 = 0$. Cujus æquationis radix maxima reperietur per Seriem recurrentem cujus ſcala relationis eſt 0, 3, − 1, unde ergo oritur, ſumtis tribus terminis initialibus pro arbitrio,

1, 1, 1, 2, 2, 5, 4, 13, 7, 35, 8, 98, − 11, &c.,

in qua Serie cum ad terminos negativos perveniatur, id indicio eſt maximam radicem eſſe negativam, eſt enim $x = -$

 $\mathit{fin.}\ 70^\circ = -0,9396926$. Quare hujus ratio in terminis initialibus eſt habenda, hoc modo

$$1-2+4-7+14-25+49-89+172-316+605-\&c.,$$

ex qua erit $y = \frac{-605}{316}$ & $x = \frac{-605}{632} = -0,957$, quæ a veritate vehementer abludit.

339. Ratio hujus diſſenſus potiſſimum eſt, quod æquationis propoſitæ radices ſint $\mathit{fin.}\ 10^\circ$, $\mathit{fin.}\ 50^\circ$, & $-\mathit{fin.}\ 70^\circ$, quarum binæ maximæ tam parum a ſe invicem diſcrepant, ut in Poteſtatibus, ad quas Seriem continuavimus, ſecunda radix $\mathit{fin.}\ 50^\circ$ adhuc notabilem teneat rationem ad radicem maximam, ideoque præ ea non evaneſcant. Hincque etiam ſaltus pendet, quod alternatim valores inventi fiant nimis magni & nimis parvi: Sic, ſumendo

$$y = \frac{-316}{172}, \text{ fit } x = \frac{-158}{172} = \frac{-79}{86} = -0,918.$$

Nam, quoniam Poteſtates radicis maximæ alternatim fiunt affirmativæ & negativæ, alternatim quoque Poteſtates ſecundæ radicis adduntur & tolluntur: quamobrem, quo hæc diſcrepantia fiat inſenſibilis, Series vehementer ulterius debet continuari.

340. Aliud vero remedium huic incommodo afferri poteſt, tranſmutando æquationem ope idoneæ ſubſtitutionis in aliam formam, cujus radices ſibi non amplius ſint tam vicinæ. Sic, ſi in æquatione $0 = 1 - 6x + 8x^3$ cujus radices ſunt $-\mathit{fin.}\ 70^\circ$, $+\mathit{fin.}\ 50^\circ$, $+\mathit{fin.}\ 10^\circ$, ponatur $x = y - 1$, æquationis $0 = 8y^3 - 24yy + 18y - 1$ radices erunt $1 - \mathit{fin.}\ 70^\circ$; $1 + \mathit{fin.}\ 50^\circ$; $1 + \mathit{fin.}\ 10^\circ$; ideoque ejus radix minima erit $1 - \mathit{fin.}\ 70^\circ$, cum tamen hæc $\mathit{fin.}\ 70^\circ$ eſſet radix maxima æquationis præcedentis; atque $1 + \mathit{fin.}\ 50^\circ$ nunc eſt radix maxima, cum $\mathit{fin.}\ 50^\circ$ ante eſſet media. Atque hoc modo quævis radix per ſubſtitutionem in maximam minimamve radicem novæ æquationis tranſmutari, ideoque per methodum hic traditam inveniri poterit.

poterit. Quia præterea in hoc exemplo radix $1 - \mathit{fin}.\, 70^\circ$ multo minor eſt, quam binæ reliquæ, etiam facile per Seriem recurrentem proxime cognoſcetur. Cap. XVII.

Exemplum IV.

Invenire radicem minimam æquationis $0 = 8y^3 - 24yy + 8y - 1$, *quæ ab unitate ſubtracta relinquet Sinum anguli* 70°.

Ponatur $y = \frac{1}{2}z$, ut ſit $0 = z^3 - 6zz + 9z - 1$, cujus radix minima invenietur per Seriem recurrentem, cujus ſcala relationis eſt $9, -6, +1$, pro radice autem maxima invenienda ſcala relationis ſumi deberet $6, -9, +1$. Pro minima ergo formetur hæc Series

$$1, 1, 1, 4, 31, 256, 2122, 17593; 145861; \&c.,$$

erit ergo proxime $z = \frac{17593}{145861} = 0,12061483$ & $y = 0,06030741$, atque $\mathit{fin}.\, 70^\circ = 1 - y = 0,93969258$, quæ a veritate ne in ultima quidem figura diſcrepat. Ex hoc ergo exemplo intelligitur quantam utilitatem idonea transformatio æquationis ope ſubſtitutionis ad inventionem radicum afferat, & quod hoc pacto methodus tradita non ſolum ad maximas minimaſve radices adſtringatur, ſed etiam omnes radices exhibere queat.

341. Cognita ergo jam quacunque æquationis propoſitæ radice proxime, ita ut, verbi gratia, numerus k quam minime a quapiam radice differat, ponatur $x - k = y$ ſeu $x = y + k$, hocque modo prodibit æquatio, cujus radix minima erit $= x - k$, quæ igitur ſi per Series recurrentes indagetur, quod facillime fiet, quia hæc radix multo minor erit, quam ceteræ, ſi ea ad k addatur habebitur radix vera ipſius x, pro æquatione propoſita. Hoc vero artificium tam late patet, ut etiamſi æquatio contineat radices imaginarias, uſum ſuum retineat.

342. Imprimis autem ſine hoc artificio radix cognoſci nequit,

LIB. I. quit, cui datur alia æqualis sed signo contrario affecta. Scilicet, si æquatio cujus maxima radix p, eadem radicem habeat $-p$, tum, etiamsi Series recurrens in infinitum continuetur, tamen radix hæc p nunquam obtinebitur. Sit, ut hoc exemplo illustremus, proposita æquatio $x^3 - x^2 - 5x + 5 = 0$, cujus maxima radix est $\sqrt{5}$, præter quam vero inest quoque $-\sqrt{5}$. Si igitur modo ante præscripto, pro radice maxima invenienda, utamur, atque Seriem recurrentem formemus ex scala relationis $1, +5, -5$, quæ erit

$$1, 2, 3, 8, 13, 38, 63, 188, 313, 938, 1563, \&c.,$$

ubi ad nullam rationem constantem pervenitur. Termini vero alterni rationem æquabilem induunt, quorum si quisque per præcedentem dividatur, reperietur quadratum maximæ radicis, sic enim est proxime $5 = \frac{1563}{313} = \frac{938}{188} = \frac{313}{63}$. Quoties ergo termini tantum alterni sese ad rationem constantem componunt, toties quadratum radicis quæsitæ proxime obtinetur. Ipsa autem radix $x = \sqrt{5}$ invenitur ponendo $x = y + 2$ unde fit $1 - 3y - 5yy - y^3 = 0$, cujus radix minima cognoscetur ex Serie

$$1, 1, 1, 9, 33, 145, 609, 2585, 10945, \&c.,$$

erit enim proxime $= \frac{2585}{10945} = 0,2361$, at $2,2361$ est proxime $= \sqrt{5}$, quæ est radix maxima æquationis.

343. Quanquam numerator fractionis, ex qua Series recurrens formatur, a nostro arbitrio pendet, tamen idonea ejus constitutio plurimum confert, ut valor radicis cito vero proxime exhibeatur. Cum enim assumtis, ut supra, Factoribus denominatoris (334.), sit terminus generalis Seriei recurrentis $= z^n (Ap^n + Bq^n + Cr^n + \&c.)$, isti coëfficientes A, B, C, &c., per numeratorem fractionis determinantur; unde fieri potest, ut A sive magnum sive parvum valorem obtineat: priori casu radix maxima p cito reperitur, posteriore vero tarde. Quin etiam numerator ita accipi potest ut A prorsus evanescat, quo

quo casu, etiamsi Series in infinitum continuetur, tamen nunquam radicem maximam p præbebit. Hoc autem evenit si numerator ita accipiatur, ut ipse eundem habeat Factorem $1 - pz$, sic enim ex computo penitus tolletur. Sic, si proponatur æquatio $x^3 - 6xx + 10x - 3 = 0$, cujus maxima radix est $= 3$, indeque formetur fractio Cap. XVII.

$$\frac{1 - 3z}{1 - 6z + 10z^2 - 3z^3}$$

ut Seriei recurrentis sit scala relationis $6, -10, +3$

$$1 + 3, 8, 21, 55, 144, 377, \&c.,$$

cujus termini prorsus non convergunt ad rationem, $1 : 3$. Eadem enim Series oritur ex fractione $\frac{1}{1 - 3z + zz}$, ac propterea maximam radicem æquationis $x^2 - 3x + 1 = 0$ exhibet.

344. Quin etiam numerator ita assumi potest, ut per Seriem recurrentem quævis radix æquationis reperiatur, quod fiet si numerator fuerit productum ex omnibus Factoribus denominatoris præter eum, cui respondet radix quam velimus. Sic, si in priori exemplo sumatur numerator $1 - 3z + zz$, fractio $\frac{1 - 3z + zz}{1 - 6z + 10z^2 - 3z^3}$, dabit hanc Seriem recurrentem $1, 3, 9, 27, 81, 243$, &c., quæ, cum sit geometrica, statim monstrat radicem $x = 3$. Fractio enim illa æqualis est huic simplici $\frac{1}{1 - 3z}$. Hinc apparet, si termini initiales, quos pro lubitu assumere licet, ita accipiantur, ut progressionem geometricam constituant, cujus Exponens æquetur uni radici æquationis, tum totam Seriem recurrentem fore geometricam, ideoque eam ipsam radicem esse exhibituram, etiamsi neque sit maxima neque minima.

345. Ne igitur, dum quærimus radicem vel maximam vel minimam, præter expectationem nobis alia radix per Seriem recurrentem exhibeatur, ejusmodi numerator debet eligi, qui cum

LIB. I. cum denominatore nullum Factorem habeat communem, quod fiet si pro numeratore unitas accipiatur, unde terminus primus Seriei erit $= 1$, ex quo solo secundum scalam relationis sequentes omnes definiantur. Hocque modo semper certe radix æquationis vel maxima vel minima, prout fuerit propositum, eruetur. Sic, proposita æquatione $y^3 * - 3y + 1 = 0$, cujus radix maxima desideratur, ex scala relationis $0, +3, -1$ incipiendo ab unitate sequens oritur Series recurrens

$$1 - 0 + 3 - 1 + 9 - 6 + 28 - 27 + 90 - 109 + 297 - 517 + 1000 - 1848 + 3517 - 6544 + \&c.,$$

quæ manifesto ad rationem constantem convergit, ostenditque radicem maximam esse negativam, atque proxime $y = \frac{-6544}{3517} = -1,860676$, quæ esse debebat $= -1,86793852$. Ratio autem supra est allata, cur tam lente ad verum valorem appropinquetur, propterea quod altera radix non multo sit minor maxima, simulque sit affirmativa.

346. His probe perpensis, quæ cum in genere tum ad exempla allata monuimus, summa utilitas hujus methodi ad investigandas æquationum radices luculenter perspicietur; artificia vero, quibus operatio contrahi, eoque promtior reddi queat, satis quoque sunt indicata; ita ut nihil insuper addendum esset, nisi casus, quibus æquatio vel radices habet æquales vel imaginarias, evolvendi superessent. Ponamus ergo denominatorem fractionis

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

habere Factorem $(1 - pz)^2$, reliquis Factoribus existentibus $1 - qz$, $1 - rz$, &c.. Seriei ergo recurrentis hinc natæ terminus generalis erit $= z^n((n+1)Ap^n + Bp^n + Cq^n + \&c.)$, quæ cujusmodi valorem sit adeptura, si n fuerit numerus vehemen-

vehementer magnus, duo casus sunt distinguendi, alter quo p est numerus major reliquis q, r, &c., alter quo p non præbet radicem maximam. Casu priori, quo p simul est radix maxima, ob coëfficientem $(n+1)$ reliqui termini $Bp^n + Cq^n$ &c., non tam cito præ eo evanescent, quam ante: sin autem q fuerit $> p$, tum quoque tarde terminus $(n+1)Ap^n$ præ Bq^n evanescet, ideoque investigatio radicis maximæ admodum evadet molesta. CAP. XVII.

EXEMPLUM I.

Sit proposita æquatio $x - 3xx + 4 = 0$, *cujus maxima radix* 2 *bis occurrit.*

Quæratur ergo maxima radix hæc modo ante exposito per evolutionem fractionis

$$\frac{1}{1 - 3z * + 4z^3}$$

quæ dabit hanc Seriem recurrentem

1, 3, 9, 23, 57, 135, 313, 711, 1593, &c.,

ubi quidem quivis terminus per præcedentem divisus dat quotum binario majorem. Cujus ratio ex termino generali facillime patet, rejectis enim in eo terminis Bp^n, Cq^n &c., erit terminus potestati z^n respondens $= (n+1)Ap^n + Bp^n$, sequens $= (n+2)Ap^{n+1} + Bp^{n+1}$, qui per illum divisus dat $\frac{(n+2)A+B}{(n+1)A+B}p > p$, nisi n jam in infinitum excreverit.

EXEM-

EXEMPLUM II.

Sit jam proposita æquatio $x^3 - xx - 5x - 3 = 0$, *cujus maxima radix* $= 3$, *reliquæ duæ æquales* $= -1$, & quæratur maxima radix ope Seriei recurrentis, cujus scala relationis est $1, +5, +3$; unde oritur

$$1,\ 1,\ 6,\ 14,\ 47,\ 135,\ 412,\ 1228,\ \&c.,$$

quæ ideo satis cito valorem 3 exhibet, quod Potestates minoris radicis -1, etiamsi multiplicentur per $n+1$, tamen mox præ Potestatibus ipsius 3 evanescant.

EXEMPLUM III.

Sin autem proponeretur æquatio $x^3 + xx - 8x - 12 = 0$, cujus radices sunt $3, -2, -2$, multo tardius maxima sese prodet. Orietur enim hæc Series

$$1,\ -1,\ 9,\ -5,\ 65,\ 3,\ 457,\ 347,\ 3345,\ 4915,\ \&c.,$$

quæ adhuc longissime continuari deberet, antequam pateret, radicem inde oriundam esse $= 3$.

347. Simili modo si tres Factores essent æquales, ita ut denominatoris Factor unus esset $(1 - pz)^3$, reliqui $1 - qz$, $1 - rz$, &c., Seriei recurrentis terminus generalis erit $=$
$$z^n\left(\frac{(n+1)(n+2)}{1.\ 2}Ap^n + (n+1)Bp^n + Cp^n + Dq^n + Cr^n \&c.\right)$$
Si ergo p fuerit maxima radix, atque n fuerit numerus tantus, ut Potestates q^n, r^n &c. præ p^n evanescant, tum ex Serie recurrente orietur radix $=$

$$\frac{\frac{1}{2}(n+2)(n+3)A + (n+2)B + C}{\frac{1}{2}(n+1)(n+2)A + (n+1)B + C}p,$$

quæ, nisi sit n numerus maximus & quasi infinitus, verum ipsius

ſius p valorem indicabit. Erit autem iſte radicis valor $= p +$ CAP. XVII.

$$\frac{(n+2)A+B}{\frac{1}{2}(n+1)(n+2)A+(n+1)B+C}p.$$

Quod ſi autem p non fuerit radix maxima, tum inventio maxime multo magis adhuc impedietur; unde ſequitur æquationes, quæ contineant radices æquales, hac methodo per Series recurrentes multo difficilius reſolvi, quam ſi omnes radices eſſent inter ſe inæquales.

348. Videamus nunc quomodo Series recurrens in infinitum continuata debeat eſſe comparata, quando denominator fractionis habet Factores imaginarios. Sint igitur fractionis

$$\frac{a+bz+cz^2+dz^3+\&c.}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\&c.}$$

Factores denominatoris reales $1-qz$, $1-rz$, &c., inſuperque Factor trinomialis $1-2pz.\,cos.\,\phi+ppzz$ continens duos Factores ſimplices imaginarios. Quod ſi ergo Series recurrens ex illa fractione orta fuerit

$$A+Bz+Cz^2+Dz^3+\ldots\ldots\ldots+Pz^n+Qz^{n+1},$$

erit, per ea quæ ſupra expoſuimus, coëfficiens $P=$

$\frac{A.\,ſin.\,(n+1)\phi+B.\,ſin.\,n\phi}{ſin.\,\phi}p^n+Cq^n+Dr^n+$ &c.. Si igitur numerus p minor fuerit, quam unus ceterorum q, r, &c., ita ut maxima radix æquationis

$$x^m-\alpha x^{m-1}-\beta x^{m-2}-\gamma x^{m-3}-\&c.=0,$$

ſit realis, tum ea per Series recurrentes æque reperietur, ac ſi nullæ radices ineſſent imaginariæ.

349. Inventio ergo maximæ radicis realis per radices imaginarias non perturbabitur, ſi hæ ita fuerint comparatæ, ut binarum, quæ Factorem realem componunt, productum non ſit

LIB. I. majus quadrato radicis maximæ. Sin autem binæ ejusmodi insint radices imaginariæ, ut earum productum adæquet vel adeo superet quadratum maximæ radicis realis, tum investigatio ante exposita nihil declarabit, propterea quod Potestas p^n, præ simili Potestate radicis maximæ nunquam evanescit, etiamsi Series in infinitum continuetur. Cujus exempla illustrationis causa hic adjicere visum est.

EXEMPLUM I.

Sit proposita æquatio $x^3 - 2x - 4 = 0$, *cujus radicem maximam investigari oporteat.*

Resolvitur hæc æquatio in duos Factores $(x - 2)(xx + 2x + 2)$; unde unam habet radicem realem 2 & duas reliquas imaginarias, quarum productum est 2, minus quam quadratum radicis realis. Quam ob rem ea per modum hactenus traditum cognosci poterit. Formetur ergo Series recurrens ex scala relationis $0, +2, +4$, quæ erit

$$1, 0, 2, 4, 4, 16, 24, 48, 112, 192, 416, 832, \&c.,$$

unde satis luculenter radix realis 2 cognosci potest.

EXEMPLUM II.

Proposita sit æquatio $x^3 - 4xx + 8x - 8 = 0$, *cujus radix una realis est* 2, *binarum imaginariarum productum vero* $= 4$, *ideoque æquale quadrato radicis realis* 2.

Quæramus ergo radicem per Seriem recurrentem, quod quo facilius fieri queat, ponamus $x = 2y$, ut habeatur $y^3 - 2yy + 2y - 1 = 0$, unde formetur Series recurrens

$$1, 2, 2, 1, 0, 0, 1, 2, 2, 1, 0, 0, 1, 2, 2, 1, \&c.,$$

in qua cum iidem termini perpetuo revertantur, nihil inde aliud

aliud colligi potest, nisi radicem maximam vel non esse realem, vel dari imaginarias, quarum productum æquale sit aut superet quadratum radicis realis. Cap. XVII.

Exemplum III.

Sit jam proposita æquatio $x^3 - 3xx + 4x - 2 = 0$, *cujus radix realis est* 1, *imaginariarum vero productum* $= 2$.

Formetur ergo ex scala relationis $3, -4, +2$, Series

$$1, 3, 5, 5, 1, -7, -15, -15, -, +1, 33, 65, 65, 1, \&c.,$$

in qua cum termini modo fiant affirmativi, modo negativi, radix realis 1 inde nullo modo cognosci poterit. Hujusmodi vero revolutiones semper ostendunt radicem, quam Series præbere debebat, esse imaginariam; hic enim radices imaginariæ potestate sunt majores quam realis 1.

350. Sit igitur in fractione generali productum binarum radicum imaginariarum pp majus quam ullius radicis realis quadratum, ita ut præ p^n reliquæ potestates q^n, r^n, &c., evanescant si n sit numerus infinitus. Hoc ergo casu fiet $P = \frac{A.\mathit{fin}.(n+1)\Phi + B.\mathit{fin}.n\Phi}{\mathit{fin}.\ \Phi}p^n$, & $Q = \frac{A.\mathit{fin}\,(n+2)\Phi + B.\mathit{fin}.(n+1)\Phi}{\mathit{fin}.\ \Phi}p^{n+1}$ ideoque $\frac{Q}{P} = \frac{A.\,\mathit{fin}.(n+2)\,\Phi + B.\,\mathit{fin}.(n+1)\,\Phi}{A.\,\mathit{fin}.(n+1)\,\Phi + B.\,\mathit{fin}.\,n\,\Phi}p$. Quæ expressio nunquam valorem constantem induet, etiamsi n sit numerus infinitus. Sinus enim Angulorum perpetuo maxime manent mutabiles, ita ut mox sint affirmativi mox negativi.

351. Interim tamen si fractiones sequentes $\frac{R}{Q}$, $\frac{S}{R}$ simili modo sumantur, indeque litteræ A & B eliminentur, simul numerus n ex calculo egredietur; reperietur enim $Ppp + R = 2Qp.\ \mathit{cof}.\Phi$, unde fit $\mathit{cof}.\ \Phi = \frac{Ppp + R}{2Qp}$; similiter vero erit

 cof. $\Phi =$

LIB. I. $cos. \Phi = \frac{Qpp + S}{2Rp}$, ex quorum duorum valorum comparatione fit $p = \surd \frac{RR - QS}{QQ - PR}$, atque $cos. \Phi = \frac{QR - PS}{2\surd(Q^2 - PR)(R^2 - QS)}$. Quam ob rem si Series recurrens jam eo usque fuerit continuata, ut præ p^n reliquarum radicum Potestates evanescant, tum hoc modo Factor trinomialis $1 - 2pz.cos.\Phi + ppzz$ poterit inveniri.

352. Quoniam iste calculus non satis exercitatis molestiam creare posset, eum totum hic apponam. Ex valore ipsius $\frac{Q}{P}$ invento oritur $AP.p.sin.(n+2)\Phi + BPp.sin.(n+1)\Phi = AQ.sin.(n+1)\Phi + BQ.sin.n\Phi$, unde fit $\frac{A}{B} = \frac{Q.sin.n\Phi - Pp.sin.(n+1)\Phi}{Pp.sin.(n+2)\Phi - Q.sin.(n+1)\Phi}$. Pari ratione erit $\frac{A}{B} = \frac{R.sin.(n+1)\Phi - Qp.sin.(n+2)\Phi}{Qp.sin.(n+3)\Phi - R.sin.(n+2)\Phi}$; æquatis his duobus valoribus fiet

$$0 = QQp.sin.n\Phi.sin.(n+3)\Phi - QR.sin.n\Phi.sin.(n+2)\Phi - PQpp.sin.(n+1)\Phi.sin.(n+3)\Phi - QQp.sin.(n+1)\Phi.sin.(n+2)\Phi + QR.sin.(n+1)\Phi.sin.(n+1)\Phi + PQpp.sin.(n+1)\Phi.sin.(n+2)\Phi.$$

Cum autem sit $sin.a\,sin.b = \frac{1}{2}.cos.(a-b) - \frac{1}{2}.cos.(a+b)$ fiet $0 = \frac{1}{2}QQp.(cos.3\Phi - cos.\Phi) + \frac{1}{2}QR.(1 - cos.2\Phi) + \frac{1}{2}PQpp.(1 - cos.2\Phi)$ quæ per $\frac{1}{2}Q$ divisa dat $(Ppp. + R)(1 - cos.2\Phi) = Qp.(cos.\Phi - cos.3\Phi)$. At est $cos.\Phi = cos.2\Phi.cos.\Phi + sin.2\Phi.sin.\Phi$ & $cos.3\Phi = cos.2\Phi.cos.\Phi - sin.2\Phi\,sin.\Phi$ unde $cos.\Phi - cos.3\Phi = 2sin.2\Phi.sin.\Phi = 4sin.\Phi^2 \times cos.\Phi$ & $1 - cos.\Phi = 2sin.\Phi^2$, ex quo erit $Ppp + R = 2Qp.cos.\Phi$, & $cos.\Phi = \frac{Ppp + R}{2Qp}$; atque $cos.\Phi = \frac{Qpp + S}{2Rp}$: unde superiores

ſuperiores valores prodeunt, ſcilicet $p = \sqrt{\frac{RR - QS}{QQ - PR}}$ & $\cos.\phi = \frac{QR - PS}{2\sqrt{(Q^2 - PR)(RR - QS)}}$.

353. Si denominator fractionis, ex qua Series recurrens formatur, plures habeat Factores trinomiales inter ſe æquales, tum, ſpectata forma termini generalis ſupra data, patebit inventionem radicum multo magis fieri incertam. Interim tamen ſi una quæcunque radix realis jam proxime fuerit detecta, tum æquationis transformatione ſemper valor ejuſdem radicis multo propior eruetur. Ponatur enim x æqualis valori illi jam detecto $+y$, atque novæ æquationis quæratur minima radix pro y, quæ addita ad illum valorem præbebit verum ipſius x valorem.

EXEMPLUM.

Sit propoſita iſta æquatio $x^3 - 3xx + 5x - 4 = 0$, *cujus unam radicem fere eſſe* $= 1$ *inde conſtat, quod, poſito* $x = 1$, *prodit* $x^3 - 3xx + 5x - 4 = -1$.

Ponatur ergo $x = 1 + y$, fietque $1 - 2y - y^3 = 0$, unde pro radice minima invenienda formetur Series recurrens, cujus ſcala relationis $2, 0, +1$, quæ erit

$$1, 2, 4, 9, 20, 44, 97, 214, 472, 1041, 2296, \&c.,$$

unde radix minima ipſius y erit proxime $\frac{1041}{2296} = 0,453397$, ita ut ſit $x = 1,453397$, qui valor tam prope vix alia methodo æque facile obtineri poterit.

354. Quod ſi autem Series quæcunque recurrens tandem tam prope ad progreſſionem geometricam convergat, tum ex ipſa lege progreſſionis ſtatim facile cognoſci poterit, cujuſnam æquationis radix ſit futura quotus qui ex diviſione unius termini per præcedentem oritur. Sint

 $P, Q,$

P, Q, R, S, T, &c., termini Seriei recurrentis a principio jam longiſſime remoti, ita ut cum progreſſione geometrica confundantur; ſitque $T = \alpha S + \beta R + \gamma Q + \delta P$, ſeu ſcala relationis $\alpha, +\beta, +\gamma + \delta$. Ponatur valor fractionis $\frac{Q}{P} = x$; erit $\frac{R}{P} = xx$; $\frac{S}{P} = x^3$ & $\frac{T}{P} = x^4$, qui in ſuperiori æquatione ſubſtituti dabunt

$$x^4 = \alpha x^3 + \beta x^2 + \gamma x + \delta.$$

unde patet quotum $\frac{Q}{P}$ tandem præbere radicem unam æquationis inventæ. Hoc vero & præcedens methodus indicat, præterea vero docet fractionem $\frac{Q}{P}$ dare maximam æquationis radicem.

355. Poteſt quoque hæc methodus inveſtigandarum radicum ſæpenumero utiliter adhiberi, ſi æquatio ſit infinita. Ad quod oſtendendum propoſita ſit æquatio $\frac{1}{2} = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} +$ &c., cujus radix minima z exhibet Arcum $30°$, ſeu Semiperipheriæ Circuli ſextantem. Perducatur ergo æquatio ad hanc formam

$$1 - 2z + \frac{z^3}{3} - \frac{z^5}{60} + \frac{z^7}{2520} - \text{\&c.} = 0.$$

Hinc ergo formetur Series recurrens, cujus ſcala relationis eſt infinita, ſcilicet

$$2,\ 0,\ -\frac{1}{3},\ 0,\ +\frac{1}{60},\ 0,\ -\frac{1}{2520},\ 0\ \text{\&c.},$$

eritque Series recurrens

$$1,\ 2,\ 4,\ \frac{23}{3},\ \frac{44}{3},\ \frac{1681}{60},\ \frac{2408}{45},\ \text{\&c.},$$

erit

erit ergo proxime $z = \frac{1681.45}{2408.60} = \frac{1681.3}{2408.4} = \frac{5043}{9632} = 0,52356$: Cap. XVII
At ex proportione Peripheriæ ad Diametrum cognita debebat esse $z = 0,523598$, ita ut radix inventa tantum parte $\frac{3}{100000}$ a vero discrepet. Hoc autem in hac æquatione commode usu venit, quod ejus omnes radices sint reales, atque a minima reliquæ satis notabiliter discrepent. Quæ conditio cum rarissime in æquationibus infinitis locum habeat, huic methodo ad eas resolvendas parum usus relinquitur.

CAPUT XVIII.

De fractionibus continuis.

356. Quoniam in præcedentibus Capitibus plura, cum de Seriebus infinitis, tum de productis ex infinitis Factoribus conflatis disserui, non incongruum fore visum est, si etiam nonnulla de tertio quodam expressionum infinitarum genere addidero, quod continuis fractionibus vel divisionibus continetur. Quanquam enim hoc genus parum adhuc est excultum, tamen non dubitamus, quin ex eo amplissimus usus in analysin infinitorum aliquando sit redundaturus. Exhibui enim jam aliquoties ejusmodi specimina, quibus hæc expectatio non parum probabilis redditur. Imprimis vero ad ipsam Arithmeticam & Algebram communem non contemnenda subsidia affert ista speculatio, quæ hoc Capite breviter indicare atque exponere constitui.

357. Fractionem autem continuam voco ejusmodi fractionem, cujus denominator constat ex numero integro cum fractione, cujus denominator denuo est aggregatum ex integro & fractione, quæ porro simili modo sit comparata, sive ista affectio in infinitum progrediatur sive alicubi sistatur. Hujusmodi ergo fractio continua erit sequens expressio

$a +$

LIB. I.

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\&c.}}}}} \quad \text{vel} \quad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\epsilon}{f+\&c.,}}}}}$$

in quarum forma priori omnes fractionum numeratores ſunt unitates, quam potiſſimum hic contemplabor, in altera vero forma ſunt numeratores numeri quicunque.

358. Expoſita ergo fractionum harum continuarum forma, primum videndum eſt, quemadmodum earum ſignificatio conſueto more expreſſa inveniri queat. Quæ ut facilius inveniri poſſit, progrediamur per gradus, abrumpendo illas fractiones primo in prima, tum in ſecunda, poſt in tertia & ita porro fractione; quo facto patebit fore

$$a = a$$

$$a+\frac{1}{b} = \frac{ab+1}{b}$$

$$a+\cfrac{1}{b+\cfrac{1}{c}} = \frac{abc+a+c}{bc+1}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{abcd+ab+ad+cd+1}{bcd+b+d}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e}}}} = \frac{abcde+abe+ade+cde+abc+a+c+e}{bcde+be+de+bc+1}$$

&c.

359. Etſi in his fractionibus ordinariis non facile lex, ſecundum quam numerator ac denominator ex litteris a, b, c, d, &c., componantur, perſpicitur, tamen attendenti ſtatim patebit, quemadmodum quælibet fractio ex præcedentibus formari queat. Quilibet enim numerator eſt aggregatum ex numeratore ultimo per novam litteram multiplicato, & ex numeratore

meratore penultimo simplici: eademque lex in denominatoribus observatur. Scriptis ergo ordine litteris $a, b, c, d,$ &c., ex iis fractiones inventæ facile formabuntur hoc modo Cap. XVIII.

$$\begin{matrix} & a & b & c & d & e \\ \frac{1}{0}; & \frac{a}{1}; & \frac{ab+1}{b}; & \frac{abc+a+c}{bc+1}; & \frac{abcd+ab+ad+cd+1}{bcd+b+d} \end{matrix}$$

ubi quilibet numerator invenitur, si præcedentium ultimus per indicem supra scriptum multiplicetur atque ad productum antepenultimus addatur; quæ eadem lex pro denominatoribus valet. Quo autem hac lege ab ipso initio uti liceat, præfixi fractionem $\frac{1}{0}$ quæ, etiamsi e fractione continua non oriatur, tamen progressionis legem clariorem efficit. Quælibet autem fractio exhibet valorem fractionis continuæ usque ad eam litteram, quæ antecedenti imminet, inclusive continuata.

360. Simili modo altera fractionum continuarum forma

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\&c.}}}}},$$

dabit, prout aliis aliisque locis abrumpitur, sequentes valores

$$a = a$$

$$a+\frac{\alpha}{b} = \frac{ab+\alpha}{b}$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}} = \frac{abc+\beta a+\alpha c}{bc+\beta}$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} = \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}$$

&c.,

 quarum

LIB. I. quarum fractionum quæque ex binis præcedentibus sequentem in modum invenietur

$$\begin{array}{ccccc} a & b & c & d & e \\ \frac{1}{0}; & \frac{a}{1}; & \frac{ab+\alpha}{b}; & \frac{abc+\beta a+\alpha c}{bc+\beta}; & \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} \\ \alpha & \beta & \gamma & \delta & \varepsilon \end{array}$$

361. Fractionibus scilicet formandis supra inscribantur indices a, b, c, d, &c., infra autem subscribantur indices α, β, γ, δ, &c.. Prima fractio iterum constituatur $\frac{1}{0}$, secunda $\frac{a}{1}$, tum sequentium quævis formabitur si antecedentium ultimæ numerator per indicem supra scriptum, penultimæ vero numerator per indicem infra scriptum multiplicetur & ambo producta addantur, aggregatum erit numerator fractionis sequentis: simili modo ejus denominator erit aggregatum ex ultimo denominatore per indicem supra scriptum, & ex penultimo denominatore per indicem infra scriptum multiplicatis. Quælibet vero fractio hoc modo inventa præbebit valorem fractionis continuæ ad eum usque denominatorem, qui fractioni antecedenti est inscriptus, continuatæ inclusive.

362. Quod si ergo hæ fractiones eousque continuentur quoad fractio continua indices suppeditet, tum ultima fractio verum dabit valorem fractionis continuæ. Præcedentes fractiones vero continuo propius ad hunc valorem accedent, ideoque perquam idoneam appropinquationem suggerent. Ponamus enim verum valorem fractionis continuæ

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\varepsilon}{e+\&c.,}}}} \quad \text{esse} = x$$

atque manifestum est fractionem primam $\frac{1}{0}$ esse majorem quam

quam x; ſecunda vero $\frac{a}{1}$ minor erit quam x; tertia $a+\frac{\alpha}{b}$ iterum vero valore erit major; quarta denuo minor, atque ita porro hæ fractiones alternatim erunt majores & minores quam x. Porro autem perſpicuum eſt quamlibet fractionem propius accedere ad verum valorem x quam ulla præcedentium; unde hoc pacto citiſſime & commodiſſime valor ipſius x proxime obtinetur; etiamſi fractio continua in infinitum progrediatur, dummodo numeratores α, β, γ, δ, &c., non nimis creſcant; ſin autem omnes iſti numeratores fuerint unitates, tum appropinquatio nulli incommodo eſt obnoxia.

363. Quo ratio hujus appropinquationis ad verum fractionis continuæ valorem melius percipiatur, conſideremus fractionum inventarum differentias. Ac, prima quidem $\frac{1}{0}$ prætermiſſa, differentia inter ſecundam ac tertiam eſt $=\frac{\alpha}{b}$; quarta a tertia ſubtracta relinquit $\frac{\alpha\beta}{b(bc+\beta)}$; quarta a quinta ſubtracta relinquit $\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma)}$, &c.. Hinc exprimetur valor fractionis continuæ per Seriem terminorum conſuetam hoc modo, ut ſit

$$x=a+\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}-\&c.,$$

quæ Series toties abrumpitur quoties fractio continua non in infinitum progreditur.

364. Modum ergo invenimus fractionem continuam quamcunque in Seriem terminorum, quorum ſigna alternantur, convertendi, ſi quidem prima littera a evaneſcat. Si enim fuerit

Lib. I.

$$x = \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\varepsilon}{f + \&c.}}}}},$$

erit per ea, quæ modo invenimus,

$$x = \frac{\alpha}{b} - \frac{\alpha\beta}{b(bc+\beta)} + \frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} - \frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\beta de+\gamma be+\delta bc+\beta\delta)} + \&c..$$

Unde, si α, β, γ, δ, &c. fuerint numeri non crescentes, uti omnes unitates, denominatores vero a, b, c, d, &c. numeri integri quicunque affirmativi, valor fractionis continuæ exprimetur per Seriem terminorum maxime convergentem.

365. His probe consideratis, poterit vicissim Series quæcunque terminorum alternantium in fractionem continuam converti, seu fractio continua inveniri cujus valor æqualis sit summæ Seriei propositæ. Sit enim proposita hæc Series

$$x = A - B + C - D + E - F + \&c.,$$

erit, singulis terminis cum Serie ex fractione continua orta comparandis

$A = \frac{\alpha}{b}$; hincque $\alpha = Ab$,

$\frac{B}{A} = \frac{\beta}{bc+\beta}$; unde fit $\beta = \frac{Bbc}{A-B}$

$\frac{C}{B} = \frac{\gamma b}{bcd+\beta d+\gamma b}$; $\gamma = \frac{Cd(bc+\beta)}{b(B-C)}$

$\frac{D}{C} = \frac{\delta(bc+\beta)}{bcde+\beta de+\gamma be+\delta bc+\beta\delta}$; $\delta = \frac{De(bcd+\beta d+\gamma b)}{(bc+\beta)(C-D)}$

&c..

At, cum sit $\beta = \frac{Bbc}{A-B}$, erit $bc+\beta = \frac{Abc}{A-B}$; unde

$\gamma =$

$\gamma = \frac{ACcd}{(A - B)(B - C)}$. Porro fit $bcd + \beta d + \gamma b =$ CAP. XVIII.

$(bc+\beta)d+\gamma b = \frac{Abcd}{A - B} + \frac{ACbcd}{(A - B)(B - C)} = \frac{ABbcd}{(A - B)(B - C)}$,

unde erit $\frac{bcd + \beta d + \gamma b}{bc + \beta} = \frac{Bd}{B - C}$ & $\delta = \frac{BDde}{(B - C)(C - D)}$;

simili modo reperietur $\varepsilon = \frac{CEef}{(C - D)(D - E)}$ & ita porro.

366. Quo ista lex clarius appareat, ponamus esse

$$P = b$$
$$Q = bc + \beta$$
$$R = bcd + \beta d + \gamma b$$
$$S = bcde + \beta de + \gamma be + \delta bc + \beta\delta$$
$$T = bcdef + \text{\&c.}$$
$$V = bcdefg + \text{\&c.},$$

erit ex lege harum expressionum

$$Q = Pc + \beta$$
$$R = Qd + \gamma P$$
$$S = Re + \delta Q$$
$$T = Sf + \varepsilon R$$
$$V = Tg + \zeta S$$
&c..

Cum igitur his adhibendis litteris sit

$$x = \frac{\alpha}{P} - \frac{\alpha\beta}{PQ} + \frac{\alpha\beta\gamma}{QR} - \frac{\alpha\beta\gamma\delta}{RS} + \frac{\alpha\beta\gamma\delta\varepsilon}{ST} - \text{\&c.},$$

367. Quoniam ergo ponimus esse

$$x = A - B + C - D + E - F + \text{\&c.},$$

erit

$$A = \frac{\alpha}{P};\ \alpha = AP$$

$$\frac{B}{A} = \frac{\beta}{Q}; \quad \beta = \frac{BQ}{A}$$
$$\frac{C}{B} = \frac{\gamma P}{R}; \quad \gamma = \frac{CR}{BP}$$
$$\frac{D}{C} = \frac{\delta Q}{S}; \quad \delta = \frac{DS}{CQ}$$
$$\frac{E}{D} = \frac{\varepsilon R}{T}; \quad \varepsilon = \frac{ET}{DR}$$
&c. &c.

Porro vero differentiis ſumendis habebitur

$$A - B = \frac{\alpha(Q - \beta)}{PQ} = \frac{\alpha c}{Q} = \frac{APc}{Q}$$
$$B - C = \frac{\alpha\beta(R - \gamma P)}{PQR} = \frac{\alpha\beta d}{PR} = \frac{BQd}{R}.$$
$$C - D = \frac{\alpha\beta\gamma(S - \delta Q)}{QRS} = \frac{\alpha\beta\gamma e}{QS} = \frac{CRe}{S}$$
$$D - E = \frac{\alpha\beta\gamma\delta(T - \varepsilon R)}{RST} = \frac{\alpha\beta\gamma\delta f}{RT} = \frac{DSf}{T},$$
&c. &c. &c.

Si bini igitur in ſe invicem ducantur, fiet

$$(A - B)(B - C) = ABcd.\frac{P}{R}; \quad \& \quad \frac{R}{P} = \frac{ABcd}{(A - B)(B - C)}$$
$$(B - C)(C - D) = BCde.\frac{Q}{S}; \quad \& \quad \frac{S}{Q} = \frac{BCed}{(B - C)(C - D)}$$
$$(C - D)(D - E) = CDef.\frac{R}{T}; \quad \& \quad \frac{T}{R} = \frac{CDef}{(C - D)(D - E)}$$
&c..

Unde, cum ſit $P = b$; $Q = \frac{\alpha c}{A - B} = \frac{Abc}{A - B}$, erit

$$\alpha = Ab$$
$$\beta = \frac{Bbc}{A - B}$$
$$\gamma = \frac{ACcd}{(A - B)(B - C)}$$

$\delta =$

$$\delta = \frac{BDde}{(B-C)(C-D)}$$

$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$

&c..

368. Inventis ergo valoribus numeratorum α, β, γ, δ, &c., denominatores b, c, d, e, &c., arbitrio noſtro relinquuntur: ita autem eos aſſumi convenit, ut, cum ipſi ſint numeri integri, tum valores integros pro α, β, γ, δ, &c., exhibeant. Hoc vero pendet quoque a natura numerorum A, B, C, &c., utrum ſint integri an fracti. Ponamus eſſe numeros integros, atque quæſito ſatisfiet ſtatuendo

$$\begin{array}{lll} b = 1 & & \alpha = A \\ c = A - B & & \beta = B \\ d = B - C & \text{unde fit} & \gamma = AC \\ e = C - D & & \delta = BD \\ f = D - E & & \varepsilon = CE \\ \&c. & & \&c. \end{array}$$

Quocirca, ſi fuerit,

$$x = A - B + C - D + E - F + \&c.,$$

idem ipſius x valor per fractionem continuam ita exprimi poterit, ut ſit

$$x = \cfrac{A}{1 + \cfrac{B}{A-B+\cfrac{AC}{B-C+\cfrac{BD}{C-D+\cfrac{CE}{D-E+\&c.}}}}}$$

369. Sin autem omnes termini Seriei ſint numeri fracti, ita ut fuerit

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \&c.,$$

habebuntur pro α, β, γ, δ, &c., ſequentes valores

$\alpha =$

LIB. I. $\alpha = \frac{b}{A}$; $\beta = \frac{Abc}{B-A}$; $\gamma = \frac{B^2 cd}{(B-A)(C-B)}$;

$\delta = \frac{C^2 de}{(C-B)(D-C)}$; $\varepsilon = \frac{D^2 ef}{(D-C)(E-D)}$; &c..

Ponatur ergo ut ſequitur

$$\begin{array}{ll} b = A; & \alpha = 1 \\ c = B - A; & \beta = AA \\ d = C - B; & \gamma = BB \\ e = D - C; & \delta = CC \end{array}$$

&c.,

eritque per fractionem continuam

$$x = \cfrac{1}{A + \cfrac{AA}{B - A + \cfrac{BB}{C - B + \cfrac{CC}{D - C + \&c.}}}},$$

EXEMPLUM I.

Transformetur hæc Series infinita

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.,$$

in fractionem continuam.

Erit ergo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c., atque, cum Seriei propoſitæ valor ſit $= l2$, erit

$$l2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \cfrac{25}{1 + \&c.}}}}}}$$

EXEMPLUM II.

Transformetur hæc Series infinita

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.,$$

ubi π denotat peripheriam circuli, cujus diameter $= 1$, in fractionem continuam.

Substitutis loco A, B, C, D, &c., numeris 1, 3, 5, 7, &c., orietur

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.,}}}}}$$

hincque, invertendo fractionem, erit

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.,}}}}$$

quæ est expressio, quam BROUNCKERUS primum pro quadratura circuli protulit.

EXEMPLUM III.

Sit proposita ista Series infinita

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \&c.,$$

quæ, ob $A = m$; $B = m + n$; $C = m + 2n$; &c., in hanc fractionem continuam mutatur

$$x = \cfrac{1}{m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}}$$

ex qua fit, invertendo,

LIB. I. $\frac{1}{x} - m = \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}$

EXEMPLUM IV.

Quoniam, ſupra §. 178., invenimus eſſe

$$\frac{\pi \, cos. \frac{m\pi}{n}}{n \, ſin. \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \&c.,$$

erit, pro fractione continuanda, $A = m$; $B = n - m$; $C = n + m$; $D = 2n - m$; &c., unde fiet

$$\frac{\pi \, cos. \frac{m\pi}{n}}{n \, ſin. \frac{m\pi}{n}} = \cfrac{1}{m + \cfrac{mm}{n-2m + \cfrac{(n-m)^2}{2m + \cfrac{(n+m)^2}{n-2m + \cfrac{(2n-m)^2}{2m + \cfrac{(2n+m)^2}{n-2m + \&c.}}}}}}$$

370. Si Series propoſita per continuos Factores progrediatur, ut ſit

$$x = \frac{1}{A} - \frac{1}{AB} + \frac{1}{ABC} - \frac{1}{ABCD} + \frac{1}{ABCDE} - \&c.,$$

tum prodibunt ſequentes determinationes

$$\alpha = \frac{b}{A};\ \beta = \frac{bc}{B-1};\ \gamma = \frac{Bcd}{(B-1)(C-1)};$$

$$\delta = \frac{Cde}{(C-1)(D-1)};\ \varepsilon = \frac{Def}{(D-1)(E-1)};\ \&c.,$$

fiat ergo ut ſequitur,

$b = A;$

$$b = A; \qquad \alpha = 1$$
$$c = B - 1; \qquad \beta = A$$
$$d = C - 1; \qquad \gamma = B$$
$$e = D - 1; \qquad \delta = C$$
$$f = E - 1; \qquad \varepsilon = D$$

&c.,

unde consequenter fiet

$$x = \cfrac{1}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + \cfrac{D}{E - 1 + \&c.}}}}}$$

EXEMPLUM I.

Quoniam, posito e numero cujus Logarithmus est $= 1$, supra invenimus esse

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \&c.,$$

seu

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \&c.,$$

hæc Series in fractionem continuam convertetur ponendo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c.: quo ergo facto habebitur

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \&c.}}}}}},$$

unde, asymmetria initio rejecta, erit

$$\frac{1}{e - 1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \&c.}}}}}$$

EXEMPLUM II.

Invenimus quoque arcus, qui radio æqualis ſumitur, coſinum eſſe $= 1 - \frac{1}{2} + \frac{1}{2.12} - \frac{1}{2.12.30} + \frac{1}{2.12.30.56} -$ &c.. Si ergo fiat $A = 1$, $B = 2$, $C = 12$, $D = 30$, $E = 56$, &c., atque Coſinus arcus qui radio æquatur, ponatur $= x$; erit

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \text{\&c.}}}}}},$$

ſeu

$$\frac{1}{x} - 1 = \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \text{\&c.}}}}}$$

371. Sit Series inſuper cum geometrica conjuncta, ſcilicet

$$x = A - Bz + Cz^2 - Dz^3 + Ez^4 - Fz^5 + \text{\&c.},$$

erit

$$\alpha = Ab;\ \beta = \frac{Bbcz}{A - Bz};\ \gamma = \frac{ACcdz}{(A - Bz)(B - Cz)};$$
$$\delta = \frac{BDdez}{(B - Cz)(C - Dz)};\ \varepsilon = \frac{CEefz}{(C - Dz)(D - Ez)};\ \text{\&c.}.$$

Ponatur nunc

$$\begin{array}{ll} b = 1; & \alpha = A \\ c = A - Bz; & \beta = Bz \\ d = B - Cz; & \gamma = ACz \\ e = C - Dz; & \delta = BDz, \end{array}$$

unde fiet

$x =$

$$x = \cfrac{A}{1 + \cfrac{Bz}{A - Bz + \cfrac{ACz}{B - Cz + \cfrac{BDz}{C - Dz + \&c.}}}}$$

372. Quo autem hoc negotium generalius abſolvamus, ponamus eſſe

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} + \frac{Dy^3}{Oz^3} - \frac{Ey^4}{Pz^4} + \&c.,$$

fietque, comparatione inſtituta,

$$\alpha = \frac{Ab}{L}; \beta = \frac{BLbcy}{AMz - BLy}; \gamma = \frac{ACM^2cdyz}{(AMz - BLy)(BNz - CMy)};$$
$$\delta = \frac{BDN^2deyz}{(BNz - CMy)(COz - DNy)}; \&c.,$$

ſtatuantur valores b, c, d, &c., ſequenti modo

$b = L;$	erit	$\alpha = A$
$c = AMz - BLy;$		$\beta = BLLy$
$d = BNz - CMy;$		$\gamma = ACM^2yz$
$e = COz - DNy;$		$\delta = BDN^2yz$
$f = DPz - EOy;$		$\varepsilon = CEO^2yz$
&c.		&c.

unde Series propoſita per ſequentem fractionem continuam exprimetur

$$x = \cfrac{A}{L + \cfrac{BLLy}{AMz - BLy + \cfrac{ACMMyz}{BNz - CMy + \cfrac{BDNNyz}{COz - DNy + \&c.}}}}$$

373. Habeat denique Series propoſita hujuſmodi formam

$$x = \frac{A}{L} - \frac{ABy}{LMz} + \frac{ABCy^2}{LMNz^2} - \frac{ABCDy^3}{LMNOz^3} + \&c.,$$

atque ſequentes valores prodibunt

$$\alpha = \frac{Ab}{L};\ \beta = \frac{Bbcy}{Mz - By};\ \gamma = \frac{CMcdyz}{(Mz - By)(Nz - Cy)};$$

$$\delta = \frac{DNdeyz}{(Nz - Cy)(Oz - Dy)};\ \varepsilon = \frac{EOefyz}{(Oz - Dy)(Pz - Ey)};$$

&c.,

ad valores ergo integros inveniendos fiat

$b = Lz;$	erit	$\alpha = Az$
$c = Mz - By;$		$\beta = BLyz$
$d = Nz - Cy;$		$\gamma = CMyz$
$e = Oz - Dy;$		$\delta = DNyz$
$f = Pz - Ey;$		$\varepsilon = EOyz$
&c.		&c.

Unde valor Seriei propositæ ita exprimetur, ut sit

$$x = \cfrac{Az}{Lz + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \&c.}}}}$$

Vel, ut lex progressionis statim a principio fiat manifesta, erit

$$\frac{Az}{x} - Ay = Lz - Ay + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \&c.}}}$$

374. Hoc modo innumerabiles inveniri poterunt fractiones continuæ in infinitum progredientes, quarum valor verus exhiberi queat. Cum enim, ex supra traditis, infinitæ Series, quarum summæ constent, ad hoc negotium accommodari queant, unaquæque transformari poterit in fractionem continuam, cujus adeo valor summæ illius Seriei est æqualis. Exempla, quæ jam hic sunt allata, sufficiunt ad hunc usum ostendendum: verumtamen optandum esset, ut methodus detegeretur, cujus beneficio, si proposita fuerit fractio continua quæcunque, ejus valor immediate inveniri posset. Quanquam enim fractio continua

tinua transmutari potest in Seriem infinitam, cujus summa per methodos cognitas investigari queat, tamen plerumque istæ Series tantopere fiunt intricatæ, ut earum summa, etiamsi sit satis simplex, vix ac ne vix quidem obtineri possit. Cap. XVIII.

375. Quo autem clarius perspiciatur, dari ejusmodi fractiones continuas, quarum valor aliunde facile assignari queat, etiamsi ex Seriebus infinitis, in quas convertuntur, nihil admodum colligere liceat, consideremus hanc fractionem continuam

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \&c.}}}},$$

cujus omnes denominatores sunt inter se æquales; si enim hinc modo supra exposito, fractiones formemus

$$\begin{matrix} 0, & 2, & 2, & 2, & 2, & 2, & 2 \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70}, & \&c.: \end{matrix}$$

Hinc autem porro oritur hæc Series

$$x = 0 + \frac{1}{2} - \frac{1}{2.5} + \frac{1}{5.12} - \frac{1}{12.29} + \frac{1}{29.70} - \&c.,$$

vel, si bini termini conjungantur, erit

$$x = \frac{2}{1.5} + \frac{2}{5.29} + \frac{2}{29.169} + \&c.,$$

vel

$$x = \frac{1}{2} - \frac{2}{2.12} - \frac{2}{12.70} - \&c..$$

Quin etiam, cum sit

$$x = \frac{1}{4} - \frac{1}{2.2.5} + \frac{1}{2.5.12} - \frac{1}{2.12.29} + \&c.$$

+

$$+\frac{1}{4}-\frac{1}{2.2.5}+\frac{1}{2.5.12}-\frac{1}{2.12.29}+\&c.,$$

erit

$$x=\frac{1}{4}+\frac{1}{1.5}-\frac{1}{2.12}+\frac{1}{5.29}-\frac{1}{12.70}\ \&c.,$$

quæ Series etiamsi vehementer convergant, tamen vera earum summa ex earum forma colligi nequit.

376. Pro hujusmodi autem fractionibus continuis, in quibus denominatores omnes vel sunt æquales, vel iidem revertuntur; ita ut ea fractio, si ab initio aliquot terminis truncetur, toti adhuc sit æqualis, facilis habetur modus earum summas explorandi. In exemplo enim proposito, cum sit

$$x=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\&c.,}}}}$$

erit $x=\frac{1}{2+x}$, ideoque $xx+2x=1$ & $x+1=\sqrt{2}$; ita ut valor hujus fractionis continuæ sit $=\sqrt{2}-1$. Fractiones vero ex fractione continua ante erutæ, continuo propius ad hunc valorem accedunt, idque tam cito, ut vix promptior modus ad valorem hunc irrationalem per numeros rationales proxime exprimendum, inveniri queat. Est enim $\sqrt{2}-1$ tam prope $=\frac{29}{70}$, ut error sit insensibilis: namque, radicem extrahendo, erit

$$\sqrt{2}-1=0,4142135623 6,$$

atque

$$\frac{29}{70}=0,4142857142 8,$$

ita ut error tantum in partibus centesimis millesimis consistat.

377. Quemadmodum ergo fractiones continuæ commodissimum suppeditant modum ad valorem $\sqrt{2}$ appropinquandi, ita indidem

indidem facillima via aperitur ad radices aliorum numerorum proxime investigandas. Ponamus hunc in finem CAP. XVIII.

$$x = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \&c.}}}}},$$

erit $x = \frac{1}{a + x}$ & $xx + ax = 1$, unde fit $x = -\frac{1}{2}a + \sqrt{(1 + \frac{1}{4}aa)} = \frac{\sqrt{(aa+4)} - a}{2}$. Hæc ergo fractio continua inserviet valori radicis quadratæ ex numero $aa + 4$ inveniendo. Hincque adeo substituendo loco a successive numeros 1, 2, 3, 4, &c., reperientur $\sqrt{5}$; $\sqrt{2}$; $\sqrt{13}$; $\sqrt{5}$; $\sqrt{29}$; $\sqrt{10}$; $\sqrt{53}$; &c., perductis scilicet his radicibus ad formam simplicissimam: erit ergo

$$\begin{array}{l} 1,\ 1,\ 1,\ 1,\ 1,\ 1, \\ \frac{0}{1},\ \frac{1}{1},\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{5},\ \frac{5}{8},\ \&c. = \frac{\sqrt{5}-1}{2} \\ 2,\ 2,\ 2,\ 2,\ 2,\ 2 \\ \frac{0}{1},\ \frac{1}{2},\ \frac{2}{5},\ \frac{5}{12},\ \frac{12}{29},\ \frac{29}{70},\ \&c. = \sqrt{2} - 1 \\ 3,\ 3,\ 3,\ 3,\ 3,\ 3 \\ \frac{0}{1},\ \frac{1}{3},\ \frac{3}{10},\ \frac{10}{33},\ \frac{33}{109},\ \frac{109}{360},\ \&c. = \frac{\sqrt{13}-3}{2} \\ 4,\ 4,\ 4,\ 4,\ 4,\ 4 \\ \frac{0}{1},\ \frac{1}{4},\ \frac{4}{17},\ \frac{17}{72},\ \frac{72}{305},\ \frac{305}{1292},\ \&c. = \sqrt{5} - 2 \\ \&c., \end{array}$$

notandum autem eo promptiorem esse approximationem, quo major fuerit numerus a: sic in ultimo exemplo erit $\sqrt{5} = 2\frac{305}{1292}$, ut error minor sit quam $\frac{1}{1292 \cdot 5473}$, ubi 5473 est denominator sequentis fractionis $\frac{1292}{5473}$.

LIB. I. 378. Hoc vero modo aliorum numerorum radices exhiberi nequeunt, nisi qui sint summa duorum quadratorum. Ut igitur hæc approximatio ad alios numeros extendatur, ponamus esse

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \&c.}}}}}},$$

erit

$$x = \cfrac{1}{a + \cfrac{1}{b+x}} = \frac{b+x}{ab+1+ax}; \text{ ideoque } axx + abx = b.$$

&

$$x = -\frac{1}{2}b \pm \sqrt{\left(\frac{1}{4}bb + \frac{b}{a}\right)} = \frac{-ab + \sqrt{(aabb + 4ab)}}{2a}.$$

Unde jam omnium numerorum radices inveniri poterunt. Sit, verbi gratia, $a = 2$, $b = 7$; erit $x = \frac{-14 + \sqrt{14.18}}{4} = \frac{-7 + 3\sqrt{7}}{2}$; at valorem ipsius x proxime exhibebunt sequentes fractiones

$$\begin{matrix} 2, & 7, & 2, & 7, & 2, & 7 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{7}{15}, & \frac{15}{32}, & \frac{112}{239}, & \frac{239}{510}, & \&c., \end{matrix}$$

Erit ergo proxime $\frac{-7 + 3\sqrt{7}}{2} = \frac{239}{510}$ & $\sqrt{7} = \frac{2024}{765} = 2,6457516$; at revera est $\sqrt{7} = 2,6457513\,1$; ita ut error minor sit quam $\frac{3}{10000000}$.

379. Progrediamur autem ulterius ponendo

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \&c.}}}}}}},$$

erit

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c+x}}} = \cfrac{1}{a + \cfrac{c+x}{bx+bc+1}} = \frac{bx+bc+1}{(ab+1)x + abc + a + c};$$

unde $(ab+1)xx + (abc + a - b + c)x = bc + 1$ atque

$x =$

$$x = \frac{-abc - a + b - c + \sqrt{((abc + a + b + c)^2 + 4)}}{2(ab + 1)};$$ CAP. XVIII.

ubi quantitas post signum radicale posita iterum est summa duorum quadratorum, neque ergo hæc forma radicibus ex aliis numeris extrahendis inservit, nisi ad quos prima forma jam suffecerat. Simili modo si quatuor litteræ a, b, c, d, continuo repetitæ denominatores fractionis continuæ constituant, tum ea plus non inserviet quam secunda, quæ duas tantum litteras continebat, & ita porro.

380. Cum igitur fractiones continuæ tam utiliter ad extractionem radicis quadratæ adhiberi queant, simul inservient æquationibus quadratis resolvendis; quod quidem ex ipso calculo est manifestum, dum x per æquationem quadraticam affectam determinatur. Potest autem vicissim facile cujusque æquationis quadratæ radix per fractionem continuam hoc modo exprimi. Sit proposita ista æquatio

$$xx = ax + b,$$

ex qua, cum sit $x = a + \frac{b}{x}$, substituatur in ultimo termino loco x valor idem jam inventus, eritque

$$x = a + \cfrac{b}{a + \cfrac{b}{x}},$$

simili ergo modo procedendo, erit per fractionem continuam infinitam

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \&c.}}},$$

quæ autem, cum numeratores b non sint unitates, non tam commode adhiberi potest.

381. Ut autem usus in arithmetica ostendatur, primum notandum est omnem fractionem ordinariam in fractionem continuam

LIB. I. tinuam converti posse. Sit enim proposita fractio $x = \frac{A}{B}$; in qua sit $A > B$; dividatur A per B, sitque quotus $= a$ & residuum C; tum per hoc residuum C dividatur præcedens divisor B, prodeatque quotus b & relinquatur residuum D, per quod denuo præcedens divisor C dividatur; sicque hæc operatio, quæ vulgo ad maximum communem divisorem numerorum A & B investigandum usurpari solet, continuetur, donec ipsa finiatur; sequenti modo

$$\begin{array}{l} B)\,A\,(a \\ \quad C)\,B\,(b \\ \qquad D)\,C\,(c \\ \qquad\quad E)\,D\,(d \\ \qquad\qquad F \text{ \&c.,} \end{array}$$

eritque per naturam divisionis

$$A = aB + C; \text{ unde } \frac{A}{B} = a + \frac{C}{B};$$

$$B = bC + D; \quad \frac{B}{C} = b + \frac{D}{C}; \quad \frac{C}{B} = \cfrac{1}{b + \frac{D}{C}}$$

$$C = cD + E; \quad \frac{C}{D} = c + \frac{E}{D}; \quad \frac{D}{C} = \cfrac{1}{c + \frac{E}{D}}$$

$$D = dE + F; \quad \frac{D}{E} = d + \frac{F}{E}; \quad \frac{E}{D} = \cfrac{1}{d + \frac{F}{E}}$$

&c. &c. &c.

hinc, sequentes valores in præcedentibus substituendo, erit

$$x = \frac{A}{B} = a + \frac{C}{B} = a + \cfrac{1}{b + \frac{D}{C}} = a + \cfrac{1}{b + \cfrac{1}{c + \frac{E}{D}}},$$

unde tandem x per meros quotos inventos a, b, c, d, &c., sequentem in modum exprimetur, ut sit

$x =$

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \text{\&c.}}}}}}$$

EXEMPLUM I.

Sit propoſita iſta fractio $\frac{1461}{59}$, quæ ſequenti modo in fractionem continuam tranſmutabitur, cujus omnes numeratores erunt unitates. Inſtituatur ſcilicet eadem operatio, qua maximus communis diviſor numerorum 59 & 1461 quæri ſolet.

```
59) 1461 (24
    118
    ---
     281
     236
     ---
      45) 59 (1
          45
          --
          14) 45 (3
              42
              --
               3) 14 (4
                  12
                  --
                   2) 3 (1
                      2
                      -
                      1) 2 (2
                         2
                         -
                         0
```

Hinc ergo ex quotis fiet

$$\frac{1461}{59} = 24 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}}.$$

EXEMPLUM II.

Fractiones quoque decimales eodem modo tranſmutari poterunt; ſit enim propoſita

$$\sqrt{2} = 1,41421356 = \frac{141421356}{100000000},$$

unde hæc operatio inſtituatur

100000000	141421356	1
82842712	100000000	2
17157288	41421356	2
14213560	34314576	2
2943728	7106780	2
2438648	5887456	2
505080	1219324	2
418728	1010160	2
&c.	209364	

Ex qua operatione jam patet omnes denominatores esse 2, atque adeo esse $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\&c.}}}}}$,
cujus expressionis ratio jam ex superioribus patet.

Exemplum III.

Imprimis vero etiam hic attentione dignus est numerus e, cujus logarithmus est $= 1$, qui est $e = 2,718281828459$, unde oritur $\frac{e-1}{2} = 0,8591409142295$, quæ fractio decimalis, si superiori modo tractetur, dabit quotos sequentes

8591409142295	10000000000000	1
8451545146224	8591409142295	6
139863996071	1408590857704	10
139312557916	1398639960710	14
551438155	9950896994	18
550224488	9925886790	22
1213667	25010204	
	&c.	

si iste calculus exactius adhuc, assumto valore ipsius e, ulterius continuetur, tum prodibunt isti quoti

$$1,\ 6,\ 10,\ 14,\ 18,\ 22,\ 26,\ 30,\ 34,\ \&c.,$$

qui, demto primo, progressionem arithmeticam constituunt, unde patet fore $e - 1$

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{\&c.}}}}}}},$$

cujus fractionis ratio ex calculo infinitesimali dari potest.

382. Cum igitur ex hujusmodi expressionibus fractiones erui queant, quæ quam citissime ad verum valorem expressionis deducant, hæc methodus adhiberi poterit ad fractiones decimales per ordinarias fractiones, quæ ad ipsas proxime accedant, exprimendas. Quin etiam, si fractio fuerit proposita cujus numerator & denominator sint numeri valde magni, fractiones ex minoribus numeris constantes inveniri poterunt quæ, etiamsi propositæ non sint penitus æquales, tamen ab ea quam minime discrepent. Hincque problema a WALLISIO olim tractatum facile resolvi potest, quo quæruntur fractiones minoribus numeris expressæ, quæ tam prope exhauriant valorem fractionis cujuspiam in numeris majoribus propositæ, quantum fieri poterit numeris non majoribus. Fractiones autem nostra hac methodo ortæ tam prope ad valorem fractionis continuæ, ex qua eliciuntur, accedunt, ut nullæ numeris non majoribus constantes dentur quæ propius accedant.

Exemplum I.

Exprimatur ratio diametri ad peripheriam numeris tam exiguis, ut accuratior exhiberi nequeat, nisi numeri majores adhibeantur. Si fractio decimalis cognita

$$3,1415926535 \text{ \&c.},$$

modo exposito per divisionem continuam evolvatur, reperientur sequentes quoti

$$3, \quad 7, \quad 15, \quad 1, \quad 292, \quad 1, \quad 1, \quad \text{\&c.},$$

ex quibus sequentes fractiones formabuntur,

$$\frac{1}{0},\ \frac{3}{1},\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{103993}{33102},\ \text{\&c.},$$

secunda fractio jam ostendit esse diametrum ad peripheriam ut

1 : 3,

LIB. I. 1 : 3, neque certe numeris non majoribus accuratius dari poterit. Tertia fractio dat rationem *Archimedeam* 7 : 22, at quinta *Metianam* quæ ad verum tam prope accedit, ut error minor sit parte $\frac{1}{113.33102}$. Ceterum hæ fractiones alternatim vero sunt majores minoresque.

EXEMPLUM II.

Exprimatur ratio diei ad annum solarem medium in numeris minimis proxime. Cum annus iste sit $365^d, 5^h, 48', 55''$, continebit in fractione annus unus $365\frac{20935}{86400}$ dies. Tantum ergo opus est ut hæc fractio evolvatur, quæ dabit sequentes quotos

4, 7, 1, 6, 1, 2, 2, 4

unde istæ eliciuntur fractiones

$\frac{0}{1}, \frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{55}{227}, \frac{63}{260}, \frac{181}{747}$, &c..

Horæ ergo cum minutis primis & secundis, quæ supra 365 dies adsunt, quatuor annis unum diem circiter faciunt, unde calendarium *Julianum* originem habet. Exactius autem 33 annis 8 dies implentur, vel 747 annis 181 dies; unde sequitur quadringentis annis abundare 97 dies. Quare, cum hoc intervallo calendarium *Julianum* inserat 100 dies, *Gregorianus* quaternis seculis tres annos bissextiles in communes convertit.

FINIS TOMI PRIMI.